Constructive Method for Goldbach Representations Yielding Generated-Family Bounds and a Prime-Selector Heuristic

Mohamed Amine Belachhab[1]

Received July 1, 2026
Accepted October 5, 2026
Electronic access November 30, 2026

Goldbach's binary problem asks whether every even integer greater than two is a sum of two primes. This paper studies a direct two-stage interval construction that, from a sufficiently large even input 2n, selects a prime p′ just below 2n−5, forms the residual M=2n−p′, and then selects a prime q′ above M. Writing q′=M+2c yields a Goldbach representation p′+q′=2n+2c. The earlier shift identity p′=p+2k, q′=q−2k+2c is retained only as an equivalent parametrization when an input representation p+q=2n is available. Using the Baker-Harman-Pintz exponent θ=0.525, together with a finite treatment below the asymptotic threshold, the iterated construction generates Ω(x^0.724375) Goldbach even integers up to x. Under an explicit unproved Cramér-type O((log x)^2) gap assumption, the two-scale argument gives c=O((log log n)^2) and Ω(x/(log log x)^2) generated values. A separate selector heuristic uses the Hardy-Littlewood singular series but does not assume that candidate events are independent as a theorem. The revised numerical experiment reports the complete distribution of S_G(n) and the minimal shift c for 900,001 inputs. Neither full Goldbach nor generated density one is proved.

Keywords: Goldbach conjecture; primes in short intervals; prime gaps; Cramér model; singular series; constructive number theory; probabilistic heuristic.

Introduction and Literature Review

Exceptional sets in the binary Goldbach problem

Hardy and Littlewood supplied the classical circle-method heuristic for binary Goldbach, including the arithmetic correction now expressed through the singular series[1]. On the unconditional side, Montgomery and Vaughan proved a power-saving estimate for the exceptional set: if E(x) counts even integers up to x that are not sums of two primes, then $E(x) \ll x^{(1-\delta)}$ for some $\delta > 0$[2]. Li obtained a later refinement of the exceptional-set estimate[3]. Matomäki studied the exceptional set in short intervals[4], Lu obtained a further exceptional-set bound[5], and Grimmelt studied Goldbach numbers in short intervals[6]. A 2026 paper of Bhowmik and Grimmelt reviews this line and adds an explicit major-arc formula and a conditional observation concerning exceptional zeros[7].

What is known from this literature is therefore much stronger than mere infinitude: Goldbach-representable even integers have density one. What remains open is the elimination of the exceptional set altogether. The present paper does not improve E(x). Its unconditional result concerns a particular iteratively generated subfamily, so its counting theorem must be compared with exceptional-set theory only to clarify scope, not as a competing density-one result.

[1] *G.S Al Jabr, Morocco*

Primes and Goldbach numbers in short intervals

The unconditional interval input used in this paper is Baker-Harman-Pintz: for all sufficiently large x, the interval $[x-x^{0.525},x]$ contains a prime[8]. Explicit interval estimates have continued to be developed; for example, Cully-Hugill and Lee give explicit prime-location estimates[9]. Recent work has also obtained conditional short-interval results directly for Goldbach numbers. Assuming the Riemann Hypothesis, Cully-Hugill and Dudek proved that every interval $(x,x+9696(\log x)^2]$ contains a Goldbach number[10], and Chirre and Hagen improved the constant 9696 to 123[11]. In a different direction, Guth and Maynard proved new zero-density estimates that imply asymptotics for primes in intervals of length $x^{(17/30+o(1))}$[12].

These statements answer different questions. Baker-Harman-Pintz gives the one-sided existence theorem used here; Guth-Maynard concerns asymptotic prime counts on a longer power scale; the 2024 and 2026 Goldbach-interval papers are conditional on RH and concern the presence of at least one Goldbach number in a nearby interval. The construction below instead applies a prime-in-interval input twice, at two different scales, to control the jump it generates.

Prime gaps and local structure

Small-gap results show that bounded gaps occur infinitely often, but they do not furnish a uniform bound for the next prime

© The National High School Journal of Science 2026

after every integer. The Goldston-Yıldırım and Goldston-Pintz-Yıldırım correlation methods initiated the modern small-gap framework[13–15]; Zhang, the Polymath project, and Maynard established and refined bounded-gap results[16–18]. Large-gap results of Maynard, Ford-Green-Konyagin-Tao, and Ford-Green-Konyagin-Maynard-Tao demonstrate that unusually long prime-free intervals also occur[19–21]. Fourier-optimization work provides another perspective on quantitative gap questions[22].

Thus a single worst-case gap function suppresses information about the range of available primes inside a search window. That distinction matters here: the deterministic part of the paper uses a worst-case interval guarantee, whereas the unit-step question is a selector problem that asks whether at least one first-stage prime has a favorable complementary residual.

Cramér-type gap heuristics

Cramér introduced a random model for primality and used it to motivate a maximal-gap scale of order (log x)^2[23]. Granville explained important arithmetic limitations of the naive independence model and refined the heuristic picture[24]. Granville and Lumley later compared short-interval heuristics with computations[25], while Banks, Ford, and Tao introduced a sieve-aware probabilistic model that more faithfully incorporates residue-class obstructions and yields heuristic information on large gaps[26].

No uniform O((log x)^2) gap theorem is known for the primes. Accordingly, every use of that scale below is labeled an assumption or heuristic. The paper's conditional result is not presented as a consequence of the prime number theorem, RH, or the Baker-Harman-Pintz theorem.

Singular series and probabilistic selector models

For binary Goldbach, Hardy-Littlewood's heuristic predicts a representation count with a multiplicative local correction depending on the prime divisors of the target even integer[1]. Modern work continues to analyze these local factors: Goldston and Suriajaya study a Cesàro mean of the prime-pair singular series[27]; Leung proves conditional Poisson point-process limits under a uniform prime-k-tuple hypothesis and obtains mean-value results for singular series[28]; and Chou, Haag, Huryn, and Ledoan relate the error term in the Hardy-Littlewood prime-pair formula to an exponential sum over primes[29].

The missing step for the selector model in this paper is not the local factor itself but a theorem controlling dependence among the different candidate complements N−r. The singular series corrects congruence biases; it does not by itself justify independence. This is why the Poisson calculation below is used only as a qualitative heuristic and is tested numerically rather than promoted to a theorem.

Position of the present work

Languasco and Zaccagnini study the number of Goldbach representations[30]. Bauer and Wang treat binary Goldbach with primes restricted to arithmetic progressions with large modulus[31], while Bauer applies a sparse-moduli large-sieve inequality to the Goldbach problem[32]. Oliveira e Silva, Herzog, and Pardi give large-scale computational verification of even Goldbach and prime gaps[33]. Platt and Trudgian study Linnik's approximation to Goldbach and related questions[34]. Friedlander, Goldston, Iwaniec, and Suriajaya specifically examine exceptional zeros in the Goldbach problem[35]. Distributional questions for primes in short and logarithmic intervals have also been developed by Montgomery-Soundararajan, Bazzanella-Languasco-Zaccagnini, and Mastrostefano[36–38], while Languasco studies a conditional short-interval exceptional-set problem[39].

Against this background, the research question here is deliberately narrower: how far can a simple two-stage residual construction be pushed from a rigorous prime-in-interval input, and what extra selector information would be required to force the unit transition 2n→2n+2? The proved contribution is a generated-family lower bound. The Cramér-scale improvement is conditional, and the density-one unit-step statement is only a model prediction.

Known Results and Heuristic Background

Baker-Harman-Pintz short-interval theorem

Baker, Harman, and Pintz prove that there exists x0 such that every x≥x0 has a prime in [x−x^0.525,x][8]. The strict one-sided statements needed by the construction follow as corollaries. For a prime strictly below y, apply the theorem at x=y−1; after enlarging the constant and threshold, this gives y−B y^0.525<p<y. For a prime strictly above y, fix A>1 and put z=y+A y^0.525. For sufficiently large y, z^0.525<A y^0.525, hence z−z^0.525>y. Applying Baker-Harman-Pintz at z gives y<p≤y+A y^0.525. After enlarging constants and thresholds, one may therefore use a common B for the strict-below and strict-above searches.

Cramér gap conjecture and model

Cramér's 1936 probabilistic model treats integers near x as independently prime with probability about 1/log x and motivates maximal gaps on the (log x)^2 scale[23]. Arithmetic correlations invalidate literal independence, and Granville's analysis explains why local congruence structure changes refined predictions[24]. In this paper the statement 'a prime occurs

© The National High School Journal of Science 2026

within A(log y)^2 on either side of y' is therefore introduced explicitly as an unproved assumption, not as Cramér's theorem.

Hardy-Littlewood Goldbach heuristic and singular series

For a large even integer N, the Hardy-Littlewood heuristic predicts a Goldbach representation count with leading factor 2C2 S(N), where C2 is the twin-prime constant and S(N) is the local correction [1]. In the normalization used below,

$$C2 := \prod_{p>2}(1 - 1/(p-1)^2),$$
$$S(N) := \prod_{p|N, p>2}(p-1)/(p-2).$$

The factor S(N) is deterministic arithmetic data. A Poisson law for the selector count would require substantially more than inserting S(N) into a first-moment estimate. Conditional results such as Leung's show how Poisson behavior can arise when a suitably uniform Hardy-Littlewood prime-tuple hypothesis is assumed [28]; no such theorem is assumed here.

Construction Method and Prime Distribution Bounds

Let 2n be a sufficiently large even integer. Set X=2n−5. Choose a prime p' in the first-stage interval and define the compressed residual

$$M := 2n - p'.$$

A second prime q' is selected above M. Since M and q' are odd, $q'-M$ is even, so there is a positive integer c with

$$q' = M + 2c,$$

and consequently

$$p' + q' = p' + (2n - p') + 2c = 2n + 2c.$$

This is the substantive two-stage construction.

Relation to the original shift parametrization

If an input Goldbach representation p+q=2n is available, the same first-stage prime can be written $p'=p+2k$ with $k=(p'-p)/2\in Z$. Since p=2n−q,

$$q - 2k = q - (p'-p) = p+q-p' = 2n-p' = M.$$

Hence $q'=q-2k+2c$. The identities in p,q,k are therefore an algebraic parametrization of the direct construction. After p' is selected, the residual M and the second prime do not depend on the particular starting representation.

Unconditional short-interval bound

By the Baker-Harman-Pintz theorem and the strict one-sided corollaries stated above, there are constants B>0 and y0 such that for every $y\geq y0$ the required prime searches can be completed within B y^θ, with θ=0.525 [8]. For sufficiently large n choose

$$p' \in (X - B\, X^\theta, X), \quad X=2n-5.$$

Then

$$5 < M = 2n-p' < 5 + B(2n-5)^\theta.$$

Let M0 be a threshold beyond which the upper-side corollary is valid. If $M\geq M0$, choose a prime q' with $M<q'\leq M+B\, M^\theta$. Writing $q'=M+2c$ gives

$$0 < c \leq (B/2)M^\theta \leq (B/2)[5+B(2n-5)^\theta]^\theta.$$

If 5<M<M0, only finitely many odd residuals occur. Let P+(M) be the least prime greater than M and define

$$C0 := \max_{5<M<M0,\ M \text{ odd}} (P+(M)-M)/2.$$

Then $c\leq C0$ in the bounded range. Since the displayed power scale tends to infinity, C0 is absorbed for sufficiently large n, yielding

$$c = O(n^{(\theta^2)}).$$

With θ=0.525, θ^2=0.275625.

Worked Numerical Example of the Construction

The following example illustrates the mechanics of the construction and the difference between an arbitrary admissible first-stage choice and a unit-step selector. It uses the normalized heuristic window G(x)=(log x)^2 only for illustration; it is not a Baker-Harman-Pintz certificate. Take 2n=200000. Then X=199995 and $G(X)\approx 148.99$. The prime $p'=199967$ lies in (X−G(X),X). It gives M=200000−199967=33. The least prime greater than 33 is $q'=37$, so $q'=M+4=M+2c$ with c=2, and

$$199967 + 37 = 200004 = 200000 + 2(2).$$

A different admissible first-stage prime, $p'=199961$, gives M=39. The least prime above 39 is 41=M+2, so c=1 and

$$199961 + 41 = 200002.$$

Thus the interval construction itself guarantees a later Goldbach even, while the selector question asks whether one can choose p' so that the later even is the immediate successor 2n+2.

© The National High School Journal of Science 2026

Fundamental Counting and Generated-Family Bound

Starting from any fixed sufficiently large even input, iterate the two-stage construction and denote the successive outputs by $2n_1 < 2n_2 < \cdots$. Each output is Goldbach by construction, and the next application depends only on the new even input. Let $a(x)$ count generated values not exceeding x.

From $c = O(n^{\theta^2})$, there is a constant C such that while $2n_j \le x$,

$$2(n_{j+1} - n_j) = 2c_j \le C\, x^{\theta^2}.$$

A deterministic covering count therefore gives

$$a(x) \gg x^{1-\theta^2}.$$

Equivalently, with $d_{\mathrm{gen}}(x) := a(x)/(x/2)$, one has $d_{\mathrm{gen}}(x) \gg x^{-\theta^2}$. For $\theta = 0.525$,

$$\mathbf{a(x) = \Omega(x^{0.724375}).}$$

This lower bound concerns only the family generated by the construction. It is not a density-one theorem for Goldbach numbers and is not a proof of binary Goldbach.

Corrected Two-Scale Cramér-Type Analysis

The fixed-exponent argument evaluates the second prime search at the compressed residual M. The same scale separation is essential under a Cramér-type hypothesis. Motivated by Cramér's model and later refinements[23–25], assume explicitly that there exist $A > 0$ and Y_0 such that every $y \ge Y_0$ has a prime in $(y - A(\log y)^2, y)$ and a prime in $(y, y + A(\log y)^2]$.

Choose p' in $(X - A(\log X)^2, X)$. Then

$$5 < M < 5 + A(\log(2n-5))^2,$$

so $M = O((\log n)^2)$. If $M \ge Y_0$, choose q' with $M < q' \le M + A(\log M)^2$; then

$$c \le (A/2)(\log M)^2 = O((\log \log n)^2).$$

If $5 < M < Y_0$, the finite set of possible residuals is bounded by a constant C_1 exactly as in the unconditional argument. Hence

$$\mathbf{c = O((\log \log n)^2).}$$

The same covering count yields the conditional bounds

$$a(x) = \Omega(x/(\log \log x)^2), \quad d_{\mathrm{gen}}(x) \gg 1/(\log \log x)^2.$$

These statements are conditional generated-family bounds; they do not imply full Goldbach or generated density one.

Limits of the Nested Max-Gap Certificate for Unit Steps

Let $G(x)$ be a nondecreasing gap function that certifies the two prime searches at the relevant scales. The first stage gives $5 < M < 5 + G(2n-5)$, and the second gives

$$2c \le G(M) \le G(5 + G(2n-5)).$$

Therefore

$$G(5 + G(2n-5)) < 4$$

is sufficient for this particular worst-case bound to force $c = 1$, because $2c$ is a positive even integer. If G is nondecreasing and unbounded, the displayed sufficient condition eventually fails. This only shows that this nested worst-case certificate stops forcing $c = 1$; it does not rule out a selector argument using the multiplicity of first-stage primes.

A Heuristic Prime-Selector Model for the Unit-Step Transition

Selector count

Let $G(x)$ be the first-stage window length and put $N = 2n + 2$. A unit step requires $q' = M + 2$ and $p' + q' = N$. Writing $q' = r$, both r and $N - r$ must be prime, and the first-stage window implies $7 < r \le 7 + G(2n-5)$. Define

$$S_G(n) := \sum_{7 < r \le 7 + G(2n-5),\ r \text{ prime}} 1_{\{N - r \text{ is prime}\}}.$$

Then $S_G(n) > 0$ is exactly the event that the allowed window contains a unit-step selector.

First-moment heuristic and local correction

A window of length G contains heuristically about $G/\log G$ prime candidates. For each candidate r, the complementary integer $N - r$ has size about N. Using the Hardy-Littlewood local factor[1], the heuristic mean is

$$\lambda(n) := 2\, C_2\, S(N)\, G(2n-5) / [\log N \cdot \log G(2n-5)].$$

The singular series accounts for congruence obstructions. It does not establish independence among the events $\{N - r \text{ is prime}\}$; modern singular-series and conditional Poisson results likewise distinguish local corrections from a full independence theorem[27–29]. Consequently the approximation $P(S_G(n) = 0) \approx e^{-\lambda(n)}$ is used only as a model.

For the normalized heuristic $G(x) = (\log x)^2$, $\lambda(n)$ is of order $C_2\, S(2n+2) \log(2n)/\log\log(2n)$, so the model predicts a decreasing zero probability. This is not a proved density-one statement.

© The National High School Journal of Science 2026

Corrected large-n singular-series illustration

At an odd prime p, averaging the factor (p−1)/(p−2) over even N gives 1+1/[p(p−2)]. Hence the product of mean local factors is 1/C2≈1.5147801281, rather than 1. For n=10^69, direct factorization gives S(2n+2)=1.4983622183666705. With natural logarithms and G(2n−5)=(log(2n−5))^2:

Quantity	Value
log(2n)	159.5715185971491
log log(2n)	5.072492214185656
S(2n+2)	1.4983622183666705
G(2n−5)	25463.0695474003
λ(n)	31.11724960392055
e^(−λ)	3.061612377665808 × 10^−14

The numerical value is illustrative only because the independence approximation is unproved.

Numerical Experiment

Method

The finite experiment evaluates every integer n from 100,000 through 1,000,000 inclusive, for 900,001 inputs, with G(x)=(log x)^2 and natural logarithms. A sieve of Eratosthenes is constructed through 2,000,100, which covers every primality query in the experiment. For each n, set N=2n+2 and enumerate every prime r with $7<r\leq 7+G(2n-5)$; S_G(n) is the number for which N−r is also prime. The singular factor S(N), the model mean λ(n), and e^(−λ(n)) are computed from the formulas above.

For the minimal shift, enumerate every odd residual M satisfying $5<M<5+G(2n-5)$ for which $p'=2n-M$ is prime. Let P+(M) be the least prime greater than M and define c(M)=(P+(M)−M)/2. The recorded quantity is c_min(n)=min_M c(M). By construction, c_min(n)=1 if and only if S_G(n)>0, providing an internal consistency check between the two computations.

Across the full range, the empirical mean of S_G(n) is 5.8146990948 and the empirical variance is 8.5190218486. There are 2,985 zero-selector inputs, a frequency of 0.3316663%. The average model mean is 5.2489892324, while the average independent-Poisson zero probability is 1.51714135%. Thus the model has the correct rough scale for the mean but substantially overpredicts zero selectors on this finite range.

Full distribution of S_G(n)

S_G(n)	Count	Percent
0	2,985	0.331666%
1	20,091	2.232331%
2	61,813	6.868103%
3	113,914	12.657097%
4	144,838	16.093093%
5	137,101	15.233428%
6	109,656	12.183986%
7	82,507	9.167434%
8	65,170	7.241103%
9	52,095	5.788327%
10	39,837	4.426328%
11	28,148	3.127552%
12	18,253	2.028109%
13	11,224	1.247110%
14	6,248	0.694221%
15	3,263	0.362555%
16	1,580	0.175555%
17	757	0.084111%
18	315	0.035000%
19	125	0.013889%
20	53	0.005889%
21	19	0.002111%
22	5	0.000556%
23	3	0.000333%
24	1	0.000111%

The distribution is concentrated around 4–7 selectors but has a visible right tail through 24 in this range. Its variance exceeds its mean, so a single Poisson law with one fixed parameter would not describe the mixture across n even before candidate dependence is considered.

Distribution of the minimal shift c

c_min	Count	Percent
1	897,016	99.668334%
2	2,976	0.330666%
3	9	0.001000%

The value c_min=1 occurs for 897,016 of the 900,001 inputs, or 99.668334%. The remaining 2,985 inputs are exactly the S_G(n)=0 cases: 2,976 have c_min=2 and 9 have c_min=3. No tested input requires $c_{min}\geq 4$. This strong finite-range concentration is empirical evidence about the chosen window; it is not an asymptotic theorem.

Comparison with Known Exceptional-Set Results

Let E(x) be the number of even $m\leq x$ that are not sums of two primes. Montgomery and Vaughan's power-saving estimate already implies that Goldbach-representable even inte-

© The National High School Journal of Science 2026

gers have density one[2]. Li and Lu obtained later exceptional-set refinements[3,5]; Matomäki studied exceptional sets in short intervals[4]; and Grimmelt studied Goldbach numbers in short intervals[6]. The 2026 treatment of Bhowmik and Grimmelt gives a current overview together with new major-arc information[7].

The theorem proved here is of a different type. The unconditional bound Ω(x^0.724375) counts only values reached by one deterministic construction. It neither improves E(x) nor approaches the strongest density statements about all Goldbach numbers. Its purpose is structural: the two-stage argument makes explicit what follows from a prime-in-interval input alone and isolates the additional selector property needed for unit steps.

Limitations

The paper has five principal limitations. First, the generated-family lower bounds concern only outputs of the construction and do not prove binary Goldbach or generated density one. Second, Baker-Harman-Pintz is asymptotic, so finite residuals must be handled separately and no explicit numerical threshold is claimed here. Third, the O((log x)^2) prime-gap input is an unproved Cramér-type assumption rather than a theorem. Fourth, the selector calculation uses a first-moment Hardy-Littlewood correction together with a Poisson independence approximation that is not justified for the candidate events in this problem. Fifth, the computation covers only $100{,}000 \le n \le 1{,}000{,}000$ with the specific normalization G(x)=(log x)^2; the observed 99.668334% unit-selector frequency and the empirical distributions do not establish any limiting density.

Open Problems and Future Directions

Residual prime-selector conjecture

The selector question can be stated directly. For a chosen window G, ask whether every sufficiently large n admits a prime

$$p' \in (2n-5-G(2n-5),\ 2n-5)$$

such that $2n-p'+2$ is prime. This remains an unproved conjectural condition. The Poisson model and the finite experiment do not prove it.

Correlation-sensitive analysis

A natural next step is a second-moment or sieve analysis of S_G(n) that tracks dependence among the complements N−r. The correlation and small-gap methods of Goldston-Yıldırım and Goldston-Pintz-Yıldırım provide relevant precedent[13–15], while the later bounded-gap literature supplies complementary sieve techniques[16–18].

Minimal-shift statistics

The computed distribution of c_min suggests studying its moments, tails, dependence on S(2n+2), and sensitivity to the first-stage window. Such questions are numerical or conjectural at present. Large-scale Goldbach and prime-gap computation provides a useful benchmark for designing those experiments[33].

Conclusion

The construction is best viewed as a direct two-stage residual procedure. From a sufficiently large even input 2n, it selects p' near $2n-5$, forms $M=2n-p'$, and selects the next controlled prime above M, thereby producing a later Goldbach even 2n+2c. The p,q,k notation is an equivalent parametrization when an initial Goldbach representation is available, but it is not the mechanism of the construction.

Using the Baker-Harman-Pintz exponent θ=0.525, with the strict one-sided searches derived as corollaries and the bounded residual range treated separately, the construction gives c=O(n^0.275625) and Ω(x^0.724375) generated values. Under an explicitly unproved Cramér-type gap hypothesis it gives c=O((log log n)^2) and Ω(x/(log log x)^2). The nested max-gap inequality is only a sufficient unit-step certificate.

The selector remains heuristic. The full computation on 900,001 inputs finds S_G(n)=0 only 2,985 times and c_min=1 in 99.668334% of cases, but the empirical zero rate is materially below the independent-Poisson prediction. That discrepancy reinforces the central unresolved point: local singular-series corrections are not a substitute for a correlation theorem. The proved contribution is therefore the generated-family analysis and the separation of worst-case interval control from the stronger selector behavior required for c=1.

Acknowledgment

The author thanks the mathematical community and the NHSJS reviewers for detailed comments that improved the precision, literature framing, and numerical presentation of the manuscript.

References

1 G. H. Hardy, J. E. Littlewood. Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. Acta Mathematica. Vol. 44, pp. 1–70, 1923. https://doi.org/10.1007/BF02403921.

2 H. L. Montgomery, R. C. Vaughan. The exceptional set in Goldbach's problem. Acta Arithmetica. Vol. 27, pp. 353–370, 1975.

© The National High School Journal of Science 2026

3 H. Li. The exceptional set of Goldbach numbers (II). Acta Arithmetica. Vol. 92(1), pp. 71–88, 2000. https://doi.org/10.4064/aa-92-1-71-88.

4 K. Matomäki. On the exceptional set in Goldbach's problem in short intervals. Monatshefte für Mathematik. Vol. 155(2), pp. 167–189, 2008. https://doi.org/10.1007/s00605-008-0561-1.

5 W. C. Lu. Exceptional set of Goldbach number. Journal of Number Theory. Vol. 130(10), pp. 2359–2392, 2010. https://doi.org/10.1016/j.jnt.2010.03.017.

6 L. Grimmelt. Goldbach numbers in short intervals. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze. Vol. 23(3), pp. 1395–1416, 2022. https://doi.org/10.2422/2036-2145.202002_001.

7 G. Bhowmik, L. Grimmelt. The exceptional set of the Goldbach problem. Analysis Mathematica, 2026. https://doi.org/10.1007/s10476-026-00191-z.

8 R. C. Baker, G. Harman, J. Pintz. The difference between consecutive primes, II. Proceedings of the London Mathematical Society. Vol. 83(3), pp. 532–562, 2001. https://doi.org/10.1112/plms/83.3.532.

9 M. Cully-Hugill, E. S. Lee. Explicit interval estimates for prime numbers. Mathematics of Computation. Vol. 91(336), pp. 1955–1970, 2022. https://doi.org/10.1090/MCOM/3719.

10 M. Cully-Hugill, A. W. Dudek. An explicit mean-value estimate for the prime number theorem in intervals. Journal of the Australian Mathematical Society. Vol. 117(1), pp. 1–15, 2024. https://doi.org/10.1017/S1446788723000113.

11 A. Chirre, M. V. Hagen. On Goldbach numbers in short intervals. Research in Number Theory. Vol. 12, Article 82, 2026. https://doi.org/10.1007/s40993-026-00767-6.

12 L. Guth, J. Maynard. New large value estimates for Dirichlet polynomials. Annals of Mathematics. Vol. 203(2), pp. 623–675, 2026. https://doi.org/10.4007/annals.2026.203.2.6.

13 D. A. Goldston, C. Y. Yıldırım. Higher correlations of divisor sums related to primes III: small gaps between primes. Proceedings of the London Mathematical Society. Vol. 95(3), pp. 653–686, 2007. https://doi.org/10.1112/plms/pdm021.

14 D. A. Goldston, J. Pintz, C. Y. Yıldırım. Primes in tuples I. Annals of Mathematics. Vol. 170(2), pp. 819–862, 2009. https://doi.org/10.4007/annals.2009.170.819.

15 D. A. Goldston, J. Pintz, C. Y. Yıldırım. Primes in tuples II. Acta Mathematica. Vol. 204(1), pp. 1–47, 2010. https://doi.org/10.1007/s11511-010-0044-9.

16 Y. Zhang. Bounded gaps between primes. Annals of Mathematics. Vol. 179(3), pp. 1121–1174, 2014. https://doi.org/10.4007/annals.2014.179.3.7.

17 D. H. J. Polymath. Variants of the Selberg sieve, and bounded intervals containing many primes. Research in the Mathematical Sciences. Vol. 1, Article 12, 2014. https://doi.org/10.1186/s40687-014-0012-7.

18 J. Maynard. Small gaps between primes. Annals of Mathematics. Vol. 181(1), pp. 383–413, 2015. https://doi.org/10.4007/annals.2015.181.1.7.

19 J. Maynard. Large gaps between primes. Annals of Mathematics. Vol. 183(3), pp. 915–933, 2016. https://doi.org/10.4007/annals.2016.183.3.3.

20 K. Ford, B. Green, S. Konyagin, T. Tao. Large gaps between consecutive prime numbers. Annals of Mathematics. Vol. 183(3), pp. 935–974, 2016. https://doi.org/10.4007/annals.2016.183.3.4.

21 K. Ford, B. Green, S. Konyagin, J. Maynard, T. Tao. Long gaps between primes. Journal of the American Mathematical Society. Vol. 31(1), pp. 65–105, 2018. https://doi.org/10.1090/jams/876.

22 E. Carneiro, M. B. Milinovich, K. Soundararajan. Fourier optimization and prime gaps. Commentarii Mathematici Helvetici. Vol. 94(3), pp. 533–568, 2019. https://doi.org/10.4171/CMH/467.

23 H. Cramér. On the order of magnitude of the difference between consecutive prime numbers. Acta Arithmetica. Vol. 2(1), pp. 23–46, 1936. https://doi.org/10.4064/aa-2-1-23-46.

24 A. Granville. Harald Cramér and the distribution of prime numbers. Scandinavian Actuarial Journal. Vol. 1995(1), pp. 12–28, 1995. https://doi.org/10.1080/03461238.1995.10413946.

25 A. Granville, A. Lumley. Primes in short intervals: Heuristics and calculations. Experimental Mathematics. Vol. 32(2), pp. 378–404, 2023. https://doi.org/10.1080/10586458.2021.1927256.

26 W. Banks, K. Ford, T. Tao. Large prime gaps and probabilistic models. Inventiones Mathematicae. Vol. 233(3), pp. 1471–1518, 2023. https://doi.org/10.1007/s00222-023-01199-0.

27 D. A. Goldston, A. I. Suriajaya. The error term in the Cesàro mean of the prime pair singular series. Journal of Number Theory. Vol. 227, pp. 144–157, 2021. https://doi.org/10.1016/j.jnt.2021.03.004.

28 S.-K. Leung. Pseudorandomness of primes at large scales. The Quarterly Journal of Mathematics. Vol. 76(1), pp. 251–263, 2025. https://doi.org/10.1093/qmath/haae069.

29 L. Chou, S. Haag, J. Huryn, A. Ledoan. The error term in counting prime pairs. Journal of Number Theory. Vol. 278, pp. 422–450, 2026. https://doi.org/10.1016/j.jnt.2025.04.009.

30 A. Languasco, A. Zaccagnini. The number of Goldbach representations of an integer. Proceedings of the American Mathematical Society. Vol. 140(3), pp. 795–804, 2012. https://doi.org/10.1090/S0002-9939-2011-10957-2.

31 C. Bauer, Y. Wang. The binary Goldbach conjecture with primes in arithmetic progressions with large modulus. Acta Arithmetica. Vol. 159(3), pp. 227–243, 2013. https://doi.org/10.4064/aa159-3-2.

32 C. Bauer. Large sieve inequality with sparse sets of moduli applied to Goldbach conjecture. Frontiers of Mathematics in China. Vol. 12(2), pp. 261–280, 2017. https://doi.org/10.1007/s11464-016-0527-x.

33 T. Oliveira e Silva, S. Herzog, S. Pardi. Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4×10^{18}. Mathematics of Computation. Vol. 83(288), pp. 2033–2060, 2014. https://doi.org/10.1090/S0025-5718-2013-02787-1.

34 D. J. Platt, T. S. Trudgian. Linnik's approximation to Goldbach's conjecture, and other problems. Journal of Number Theory. Vol. 153, pp. 54–62, 2015. https://doi.org/10.1016/j.jnt.2015.01.008.

35 J. B. Friedlander, D. A. Goldston, H. Iwaniec, A. I. Suriajaya. Exceptional zeros and the Goldbach problem. Journal of Number Theory. Vol. 233, pp. 78–86, 2022. https://doi.org/10.1016/j.jnt.2021.06.004.

36 H. L. Montgomery, K. Soundararajan. Primes in short intervals. Communications in Mathematical Physics. Vol. 252, pp. 589–617, 2004. https://doi.org/10.1007/s00220-004-1222-4.

37 D. Bazzanella, A. Languasco, A. Zaccagnini. Prime numbers in logarithmic intervals. Transactions of the American Mathematical Society. Vol. 362(5), pp. 2667–2684, 2010. https://doi.org/10.1090/S0002-9947-09-05009-0.

38 D. Mastrostefano. Short intervals containing a prescribed number of primes. Journal of Number Theory. Vol. 197, pp. 49–61, 2019. https://doi.org/10.1016/j.jnt.2018.06.013.

39 A. Languasco. A conditional result on the exceptional set for Hardy-Littlewood numbers in short intervals. International Journal of Number Theory. Vol. 5(6), pp. 933–951, 2009. https://doi.org/10.1142/S179304210900247X.

© The National High School Journal of Science 2026