SQTPP: Shifted Quadratic Twice Prime Power Equation – Necessary Conditions and Structural Analysis

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We study the Diophantine equation $n^2+1=2p^k$ in positive integers with p prime and $k\geq 1$. It lies at the intersection of two classical families, the Lebesgue-Nagell equation and the negative Pell equation, and the results bearing on it are scattered across the literature on generalized Lebesgue-Ramanujan-Nagell equations. The objective is a single classification organized by the exponent k, identifying what remains open. Necessarily conditions are derived by elementary modular arithmetic and, independently, by factoring the left-hand side in the Gaussian integers $\mathbb{Z}[i]$. The case $k=2$ is reduced to the negative Pell equation through the continued fraction expansion of $\sqrt{2}$, and the case $k>3$ are settled using established theorems on perfect powers in the Pell sequence. All reported solutions were verified by direct computation. Every solution triple satisfies three necessary conditions: is odd, $p>2$, and $p\equiv 1 \pmod 4$. Writing $n=2r+1$ gives $p=1+2r(r+1)$ for $k=1$; for every p, $n=\sqrt{2p^k-1}$ must be an integer. The prime power is always a sum of two consecutive squares. For $k=1$ the problem is an analogue of Landau's fourth problem; for $k=2$ it is known that the negative Pell equation has infinitely many solutions, and only $k=2$ is classified completely (by the Pell sequence being $(239,13,4)$. Only $k=1$ and $k=2$ remain open, and we conjecture that $k=2$ admits infinitely many solutions with p prime.

Keywords Diophantine equations; negative Pell equation; Gaussian integers; Pell numbers; Newman-Shanks-Williams numbers; Lebesgue-Ramanujan-Nagell equations.

Introduction

A central theme in number theory has always been the study of the integer solutions to polynomial equations, especially Diophantine equations of the form $x^2+D=By^k$. The Lebesgue-Nagell equation $x^2+D=y^n$, with D as a fixed integer and n as a variable exponent, has been solved for many specific values of D, and the range of values now settled has been extended considerably by recent work[1–3]. The principal tools are primitive divisor theorems for Lucas and Lehmer sequences, which bound the variable exponent, together with modular methods based on the Galois representations attached to Frey curves[4–7]. Classical descent in quadratic fields settles many further individual cases[8–10]. The closely related Pell equation $x^2-Dy^2=1$ and the negative Pell equation $x^2-Dy^2=-1$ have been studied extensively[11,12]. For non-square D , the positive equation always has infinitely many solutions parametrized by the continued fraction expansion of $\sqrt{D}$; the negative equation, by contrast, is solvable only for those D whose continued fraction period is odd, and for most D it has no solutions at all[11]. The case $D=2$, which is the one relevant to this paper, has odd period, so its negative equation is solvable with complete solutions parametrized by the continued fraction expansion[11] of $\sqrt{2}$. This work studies the hybrid of the two families of equations obtained by doubling the prime power.

$$n^2+1=2p^k$$

We call this equation Shifted Quadratic Twice Prime Power Equation: "shifted" because the quadratic term is incremented by one, and "twice" because the right-hand side carries the factor 2, forcing the right side to be even and the left-hand side to factor as $(n+i)(n-i)$ in the Gaussian integers[11] $Z[i]$. In this equation $p>2$ is prime and $k,n\geq 1$ are positive integers. The restriction that the base be prime narrows the equation to make the primality structure of solutions interesting and unique. The case $k=2$ is the negative Pell equation $n^2-2p^2=-1$, whose solutions (n,p) consist of the NSW numbers paired with the odd-indexed Pell numbers[11,13,14]. The case $k=4$ is the well-known Ljunggren equation $x^2+1=2y^4$, solved by Ljunggren in 1942, with a modern proof given by Steiner and Tzanakis[15,16] and an elementary proof given by Tao[17]. The case $k=1$ connects to Landau's fourth problem on the infinitude of primes of the form m^2+1; questions about the prime factorization of numbers of the form m^2+1 have a long history going back to Störmer and Lehmer[18].

The equation studied here belongs to a well-developed

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line of research on generalized Lebesgue–Ramanujan–Nagell equations of the form $x^2 + D = 2y^n$. Cohn[19] solved the base case $x^2 + 1 = 2y^q$ for odd prime exponents q by classifying the perfect powers in the Pell sequence; Pethő[20] obtained the classification of perfect Pell powers independently, and Bennett[21] later treated perfect powers in Pell-type sequences by different methods. Tengely[22] resolved broad families of the equations $x^2 + a^2 = 2y^p$ and $x^2 + q^{2m} = 2y^p$, showing in the latter case[23] that the exceptional solutions are exactly those in which y is a sum of two consecutive squares. Pink and Tengely[24] studied full powers in the arithmetic progressions a^2, y^n, x^2, which is equivalent to $x^2 + a^2 = 2y^n$, and gave explicit bounds for the exponent n; Pink[25] extended these bounds to $x^2 + (p_1^{z_1} \cdots p_s^{z_s})^2 = 2y^n$. Abu Muriefah, Luca, Siksek, and Tengely[26] gave sharp exponent bounds for $x^2 + C = 2y^n$ with $C \equiv 1 \pmod 4$ using the primitive divisor theorem for Lehmer sequences, and Zhu, Le, and Togbé[27] described all solutions of $x^2 + p^{2m} = 2y^n$. Surveys of this literature are given by Bérczes and Pink[28] and by Le and Soydan[29]; further representative results concern neighbouring families of the same type. For a fixed constant term, the equation has been settled for many individual constants by classical descent, and the resulting body of work has been surveyed[8–10,30]. When the constant term is instead built from a few fixed small primes carrying variable exponents, several such equations have been resolved completely[31–35]. A third group of results treats single equations with one specific constant, together with variants carrying a leading coefficient[36–38]. Finally, the number of solutions of the generalized Ramanujan-Nagell equation has been bounded, and the perfect powers in products of Pell and Pell-Lucas numbers have been determined[39,40]. Positioned against this body of work, what remains open for the present equation is the $k = 2$ primality question, and the contribution of this paper is the elementary unification of the known results into a single classification organized by the exponent k.

The contributions of this paper are three-fold:

1. A unified derivation of the necessary conditions that n is odd, $p > 2$, and $p \equiv 1 \pmod 4$. We present two proofs of the last condition: proof via Euler's criterion, and a proof through factorization[11] in $Z[i]$.

2. A structural identity holding for every $k \geq 1$: converting $n = 2m + 1$ transforms the equation into $p^k = m^2 + (m + 1)^2$, so every solution forces p^k itself to be a sum of two consecutive squares. We state and prove this identity in full generality (Proposition 1). This is exactly the mechanism for Cohn's proof[19] and of Tengely's theorem on $x^2 + q^{2m} = 2y^p$, where solutions with y (a sum of two consecutive squares) form the exceptional family.

3. A classification of solutions by k. For $k = 1$, the equation is analogous to Landau's fourth problem of infinite primes of the form $m^2 + 1$. For $k = 2$, it reduces to the negative Pell's equation $n^2 - 2p^2 = -1$ with the conjecture[13,14] that there are infinitely many solutions to this equation with prime p. For all $k \geq 3$, we obtain a complete classification (Theorem 1): combining Cohn's theorem with Ljunggren's theorem, the only nontrivial solution triple with $k \geq 3$ is $(239, 13, 4)$ and there are no solutions with $p > 1$ for any other exponent $k \geq 3$.

The rest of the paper proves the necessary conditions and the consecutive-squares identity. It explains the conditions for each case $k = 1, 2, \geq 3$ in turn and states the central conjecture for the $k = 2$ case. Throughout, results quoted from the literature are labelled as facts and carry their source; the lemmas, propositions, corollary and theorem are proved here. Theorem 1 being an assembly of the quoted facts rather than a new finiteness result.

Methods

Research design and analytical tools

This study is theoretical and mathematical rather than experimental. It involves no human or animal subjects and no collected data set, so no ethical approval was required. The research design is a structured classification of the solution set of the equation by the exponent k, with each case analysed by the tool appropriate to it. Four classes of tool are used: elementary modular arithmetic, which yields the necessary conditions of Lemmas 1-3; unique factorization in the Gaussian integers $Z[i]$, which gives an independent second proof of the congruence condition on p; the continued fraction expansion of $\sqrt{2}$, which parametrizes the negative Pell equation and therefore the case $k = 2$; and established classification theorems for perfect powers in linear recurrence sequences, which settle the cases $k \geq 3$.

Every solution triple reported in this paper, and every entry of Tables 1-3, was verified by direct arithmetic computation of both sides of the equation together with a primality check of p; the Pell and Newman-Shanks-Williams values were generated from their standard recurrences. Throughout, statements quoted from the literature are labelled as facts and carry their source, while the definitions, lemmas, propositions, corollary, remark and theorem stated below are proved here.

Preliminaries

The proofs in this paper combine classical tools in elementary modular arithmetic; the continued fraction parametrization of the negative Pell equation; and two finiteness theorems for perfect powers in linear recurrence sequences. The Gaussian integers $Z[i] = \{a + bi : a, b \in Z\}$ form a number system in which, exactly as in the ordinary integers, every element

factors uniquely into primes up to units[11]; it is this unique factorization that allows the identity $n^2+1=(n+i)(n-i)$ to be exploited multiplicatively. A primitive divisor theorem, such as that of Bilu, Hanrot, and Voutier[4], states that beyond a small initial segment every term of a Lucas or Lehmer recurrence sequence has a prime factor that divides no earlier term; theorems of this type help prove the family of equations[23,26,27] $x^2+C=2y^n$.

Definition 1 (Solution Triple)

A triple $(n,p,k) \in^+$ is a solution if p is prime, $k \geq 1$ and $n^2+1=2p^k$.

Definition 2 (Quadratic Residue)

An integer a is a quadratic residue modulo p if $x^2 \equiv a \pmod{p}$ for some integer x. By Euler's criterion, for odd prime p with $p \nmid a$, this holds iff $a^{\left(\frac{p-1}{2}\right)} \equiv 1 \pmod{p}$.

Fact 1 (Euler's Criterion)

For an odd prime p the integer -1 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod 4$. This follows because $(-1)^{\frac{p-1}{2}}=1$ iff $\frac{(p-1)}{2}$ is even, i.e., iff $4 \mid (p-1)$.

Fact 2 (Negative Pell's Equation for D=2)[11]

All positive integer solutions to $x^2-2y^2=-1$ are given by $x_m+y_m\sqrt{2}=(1+\sqrt{2})^{(2m-1)}$ for $m=1,2,3\ldots$ yielding the pairs $(1,1),(7,5),(41,29),(239,169),(1393,985),\ldots$. In particular, the equation has infinitely many positive integer solutions.

Fact 3 (Cohn[19]; Pethő[20])

The only Pell number with $n \geq 1$ that is a perfect power greater than 1 is $P_7=169=13^2$. Consequently, the only solution of $x^2+1=2y^q$ in positive integers for an odd prime exponent q is $x=y=1$.

Fact 4 (Ljunggren[15]; Steiner–Tzanakis[16]; Tao[17])

The only solutions of $x^2+1=2y^4$ in positive integers are $(x,y)=(1,1)$ and $(x,y)=(239,13)$.

The theorems and facts stated here are for reference; their proofs lie outside the elementary methods used in this paper.

Results

Necessary Conditions

The following three lemmas use modular arithmetic to establish the necessary conditions for the solution triple.

Lemma 1: If (n,p,k) is a solution triple, then n is odd.

Proof: Suppose n is even. Then $n^2 \equiv 0 \pmod 4$, so $n^2+1 \equiv 1 \pmod 4$. If p is odd , then $2p^k \equiv 2 \pmod 4$; if $p=2$, then $2p^k=2^{k+1} \equiv 0 \pmod 4$ for $k \geq 1$, both 0 (mod 4) and 2 (mod 4) not equal to 1 (mod 4) , a contradiction. Hence n must be odd.

Lemma 2: If (n,p,k) is a solution triple, then $p>2$.

Proof: Suppose $p=2$. Then $2p^k=2^{k+1} \equiv 0 \pmod 4$ for $k \geq 1$. By Lemma 1, n is odd, so $n^2+1 \equiv 2 \pmod 4$. As $2 \not\equiv 0 \pmod 4$, this is a contradiction. Hence $p>2$.

Lemma 3: If (n,p,k) is a solution triple, then $p \equiv 1 \pmod 4$.

First proof (via Euler's criterion): Since $p \mid n^2+1$, we have $n^2 \equiv -1 \pmod p$, so -1 is a quadratic residue modulo p. By Lemma 2, p is odd, and by Fact 1 it follows that $p \equiv 1 \pmod 4$.

Second Proof (via Gaussian integers): In the Gaussian integers we have the factorization

$$n^2+1=(n+i)(n-i)=2p^k, \quad k \geq 1$$

This implies $p \mid (n+i)(n-i)$ in $Z[i]$. By Lemma 2, the prime p is odd. Recall that a rational prime $q \equiv 3 \pmod 4$ remains prime[11] in $[i]$. If $p \equiv 3 \pmod 4$, then p is a Gaussian prime, and dividing the product $(n+i)(n-i)$ it would divide one of the factors, say $p \mid n+i$. Writing $n+i=p(a+bi)$ with $a,b \in$ and comparing imaginary parts gives $pb=1$, which is impossible for $p>1$. The same holds for $n-i$. Hence $p \not\equiv 3 \pmod 4$, and since p is odd, we conclude $p \equiv 1 \pmod 4$.

These lemmas together characterize the solution space in which the solution triple must lie. These are necessary but not sufficient conditions for the solution triple. Not every odd n and prime $p \equiv 1 \pmod 4$ gives a solution to the equation.

Proposition 1 (Consecutive-squares Identity for every $k \geq 1$):

Let $k \geq 1$. A pair of positive integers (n,p) satisfies $n^2+1=2p^k$ if and only if n is odd and, writing $n=2m+1$ with $m \geq 0$,

$$p^k=m^2+(m+1)^2.$$

In particular, in every solution triple the prime power p^k is a sum of two consecutive squares[41].

Proof: If $n^2+1=2p^k$ then n is odd by (the argument of) Lemma 1. Substituting $n=2m+1$,

$$\begin{aligned}(2m+1)^2+1=2p^k &\Leftrightarrow 4m^2+4m+2=2p^k \\ &\Leftrightarrow 2m^2+2m+1=p^k \\ &\Leftrightarrow m^2+(m+1)^2=p^k,\end{aligned}$$

and each step is reversible.

Proposition 2 (An analogue of Landau's fourth problem for $k=1$):

For $k=1$, the equation becomes analogous to Landau's fourth problem, the conjecture that there are infinitely many primes of the form m^2+1.

If $n^2+1=2p$, then by Proposition 1 with $k=1$, the solutions with p prime correspond exactly to primes of the form

Table 1 Example solutions of $n^2+1=2p$, each satisfying n odd, $p>2$ prime, and $p\equiv 1 \pmod 4$ based on direct arithmetic computation.

n	p	k	Verification
3	5	1	$3^2+1=10=2\times 5;\ 5\equiv 1 \pmod 4$
9	41	1	$9^2+1=82=2\times 41;\ 41\equiv 1 \pmod 4$
11	61	1	$11^2+1=122=2\times 61;\ 61\equiv 1 \pmod 4$

$p=m^2+(m+1)^2=2m^2+2m+1$, the sums of two consecutive squares[41]. The infinitude of such primes is an open problem of the same character as Landau's fourth problem. Questions about the prime divisors of the closely related numbers m^2+1 go back to Störmer and were investigated by Lehmer[18]. Table 1 lists some of the solutions of the equation that satisfy these conditions.

Proposition 3: Reduction to the Negative Pell Equation for $k=2$

For the case $k=2$, the proposed equation reduces to a classical equation whose solution set is well understood. A pair (n,p) gives a solution triple $(n,p,2)$ if and only if (n,p) is a positive integer solution to

$$x^2-2y^2=-1$$

with $y=p$ prime.

Proof: The equation $n^2+1=2p^2$ can be rearranged to $n^2-2p^2=-1$ which is the negative Pell's equation.

Proposition 4 (Infinite solutions of the Negative Pell's Equation): The equation $x^2-2y^2=-1$ has infinitely many positive integer solutions.

Proof: The continued fraction expansion of $\sqrt{2}$ is $[1;2,2,2,\ldots]$. Let h_m and k_m be the m-th convergent, where $m=0,1,2,\ldots$ Then,

$$h_m^2-2k_m^2=(-1)^{m+1}$$

Therefore, for every even m,

$$h_m^2-2k_m^2=-1$$

Since there are infinitely many non-negative even integers m, there are infinitely many positive integer solutions to

$$x^2-2y^2=-1$$

Corollary 1: There are infinitely many positive integer solution pairs (n,p) to $n^2+1=2p^2$, namely the pairs listed in Fact 2. Each pair for which p is prime gives a solution triple with $k=2$, for example $(7,5)$ and $(41,29)$.

Primality of Solutions

Corollary 1 shows infinitely many candidates for solution triples. The question that remains is whether infinitely many p values are prime. Among the first six negative Pell's solutions, $p\in\{1,5,29,169,985,5741\}$. Of these, $5,29$ and 5741 are prime, while $1,169$ and 985 are not prime. Proving infinitely many are prime is like the open problem of whether n^2+1 is prime for infinitely many n. The central claim is stated as a conjecture:

Table 2 Primality of the odd-indexed Pell numbers P_{2j-1} for $1\le 2j-1\le 41$, computed directly. Each prime entry yields a solution triple $(n,p,2)$ with $n^2=2p^2-1$.

Index 2j-1	$p=P(2j-1)$	Prime?	*n* (NSW number)
1	1	No (unit)	1
3	5	Yes	7
5	29	Yes	41
7	169	No (13^2)	239
9	985	No (5·197)	1393
11	5741	Yes	8119
13	33461	Yes	47321
15	195025	No	275807
29	44560482149	Yes	63018038201
41	1746860020068409	Yes	2470433131948081

Conjecture 1 (Shifted Quadratic Twice Prime Power Conjecture): There are infinitely many positive integers n such that $n^2+1=2p^2$ with p prime.

By Proposition 3 and Fact 2 this is not an independent conjecture but a restatement of an existing open problem: the admissible values of p are exactly the odd-indexed Pell numbers P_{2j-1} (OEIS A001653), the corresponding n are the Newman-Shanks-Williams numbers (OEIS A002315), and the conjecture asserts that infinitely many odd-indexed Pell numbers are prime. Whether the Pell sequence contains infinitely many primes is open, and the indices of the known Pell primes are catalogued as OEIS A096650[13]. The conjecture is stated here for completeness and to place the present classification as a summary of all the prior work.

Six of the first twenty-one odd-indexed Pell numbers are prime; the count of prime Pell numbers is expected to grow without bound but no proof is known. Table 2 is offered as evidence consistent with Conjecture 1, not as support for it.

Theorem 1 (Complete classification for $k\ge 3$):

The only solution triple (n,p,k) of $n^2+1=2p^k$ with $k\ge 3$ is $(n,p,k)=(239,13,4)$. In particular, for every $k\ge 3$ with $k\neq 4$ the equation has no solution with p prime, and for $k=4$ it has exactly one.

Proof: Let (n,p,k) be a solution with $k\ge 3$. Every integer $k\ge 3$ either has an odd prime divisor or is divisible by 4.

Case 1: k has an odd prime divisor q. Write $k=qs$. Then $n^2+1=2(p^s)^q$, so (n,p^s) is a positive solution of $x^2+1=2y^q$ with odd prime exponent q. By Fact 3 (Cohn[19]), the only such solution is $x=y=1$, forcing $p^s=1$, which is impossible for a prime p.

Case 2: k has no odd prime divisor, so k is a power of 2 with

Table 3 Example solution triples of $n^2+1=2p^k$, each satisfying n odd, $p>2$ prime, and $p\equiv 1 \pmod 4$ based on direct arithmetic computation.

n	p	k	Verification
7	5	2	$7^2+1=50=2\times 5^2;\ 5\equiv 1 \pmod 4$
41	29	2	$41^2+1=1682=2\times 29^2;\ 29\equiv 1 \pmod 4$

$k\geq 4$; in particular $4\mid k$. Write $k=4t$. Then $n^2+1=2(p^t)^4$, so (n,p^t) solves $x^2+1=2y^4$. By Fact 4 (Ljunggren[15,16]), either $p^t=1$, which is impossible, or $(n,p^t)=(239,13)$, which forces $t=1$, $p=13$, $k=4$, and $n=239$.

The trivial integer solution $n=1$, $p=1$ (for any k) is excluded throughout because $p=1$ is not prime. Theorem 1 shows that the paper's classification is global across every exponent: the only nontrivial solution triple with $k\geq 3$ is $(239,13,4)$, and $k=1$ and $k=2$ are the only cases in which the existence of infinitely many solutions remains open.

Solution Triples Verification

The two $k=2$ triples below are verified by direct arithmetic and satisfy all necessary conditions of Lemmas 1-3 (Table 3); the remaining triple $(239,13,4)$ is the unique $k\geq 3$ solution of Theorem 1 and satisfies $239^2+1=57122=2\cdot 13^4$ with $13\equiv 1 \pmod 4$.

Triples $(7,5,2)$ and $(41,29,2)$ arise from the second and third solutions of the negative Pell equation listed in Fact 2, i.e. from the odd powers of $1+\sqrt{2}$; their p values 5 and 29 are the odd-indexed Pell numbers $P_3=5$ and $P_5=29$.

Discussion

Interpretation and attribution of the classification

The classification in Theorem 1 rests entirely on Cohn[19] (with Pethő's independent proof[20] and Tao's elementary proof[17] as alternatives) and on Ljunggren[15,16]. The primitive divisor theorem of Bilu, Hanrot, and Voutier[4] and the modular methods of Bennett and Skinner[5] are the general tools by which such stronger results on the family $x^2+C=2y^n$ have been obtained[23,26,27].

Remark 1 (why $k\geq 3$ is a constrained-power problem in the Pell sequence): The triple $(239,13,4)$ does not lie outside the Pell equations; it lies inside them under an extra constraint. Writing $n^2+1=2p^4$ as $n^2-2(p^2)^2=-1$ exhibits it as a solution of the negative Pell equation whose second coordinate happens to be a perfect square, so $239=\mathrm{NSW}_4$ and $P_7=169=13^2$. The same holds in general: for even k, a solution forces the Pell coordinate P_{2j-1} to be a perfect $k/2$ power, while for k with an odd prime divisor q the equation is a perfect q power condition on the same sequence. Both are instances of the perfect-powers-in-the-Pell-sequence problem, and it is precisely Cohn's classification of those powers[19] that settles $k\geq 3$. Tao[17] states the resulting classification directly (his Theorem 5).

Proposed Variants:

Five variants are proposed to guide further research. Each generalizes a single parameter of the original equation.

Type A (Varied Addend):

$n^2+d=2p^k$ for $d\geq 1$ with p an odd prime. Then $n\equiv d \pmod 2$, i.e., n and d have the same parity, and if $p\nmid d$ then $-d$ must be a quadratic residue modulo p. The right side $2p^k$ is even, so $n^2\equiv -d\equiv d \pmod 2$, giving the parity claim and reducing the equation modulo p gives $n^2\equiv -d \pmod p$. The family $x^2+a^2=2y^p$, the case $d=a^2$, is resolved in Tengely[22] and bounded in Pink and Tengely[24,25]; related cases in which d is prime power are treated in references[23,27].

Type B (Varied Multiplier):

$n^2+1=cp^k$ for $c\geq 3$. If c is odd and p is odd, the right side is odd, forcing n even; and for $p\nmid 1$ trivially, reduction modulo p still gives $n^2\equiv -1 \pmod p$, so the condition $p\equiv 1 \pmod 4$ of Lemma 3 persists for every odd prime p dividing the right side. *Proof:* cp^k odd implies n^2+1 odd, so n^2 is even and n is even; the residue statement is Fact 1 applied as in the first proof of Lemma 3.

Type C (Combined Type A and Type B):

$n^2+d=cp^k$. The most general single-prime variant; the analysis is specific to each pair (c,d). The survey of Le and Soydan[29] organizes what is known for the wider family.

Type D (Coefficient of n):

$an^2+1=2p^k$ for $a\geq 2$. For $k=2$, it yields $an^2-2p^2=-1$, a generalized Pell's equation whose solvability depends on a. The closely related equation $2x^2+1=y^n$ is completely solved by Tao[17], and Ljunggren[37] treated the general form $Cx^2+D=2y^n$.

Type E (Higher exponent of n):

$n^j+1=2p^k$ for $j\geq 3$. The $j=3$ case becomes a shifted cubic power problem. For $j,k\geq 2$ such questions concern two nearly equal perfect powers and are in the spirit of Catalan's conjecture, resolved by Mihailescu[42].

Limitations

Three limitations should be noted. First, this paper establishes necessary conditions rather than sufficient ones: not every odd n paired with a prime $p\equiv 1 \pmod 4$ yields a solution, so Lemmas 1-3 narrow the search space without characterizing

it. Second, the complete classification for $k \geq 3$ is not proved here. It is assembled from Cohn's classification of perfect Pell powers and from Ljunggren's theorem, both of which rest on methods well outside the elementary techniques used in this paper, and Theorem 1 should therefore be read as a synthesis of existing results rather than as a new finiteness result. Third, Conjecture 1 remains unproved, and the computation summarized in Table 2 covers only the odd-indexed Pell numbers of index at most 41. Because the prime values in such a sequence thin rapidly, a finite table of this kind is consistent with the conjecture but cannot constitute evidence for it; extending the computation would sharpen the empirical picture without changing its logical status.

Conclusion

We have studied the Diophantine equation $n^2+1=2p^k$ in this paper, the Shifted Quadratic Twice Prime Power Equation. Three necessary conditions for every solution triple of the Shifted Quadratic Twice Prime Power Equation follow from Lemmas 1-3: n must be odd, p must exceed 2, and $p \equiv 1 \pmod 4$. For $k=1$, the equation is analogous to Landau's fourth problem (Proposition 2). For $k=2$, the equation reduces to the negative Pell equation $x^2-2y^2=-1$ (Proposition 3), which possesses infinitely many solutions (Fact 2, Corollary 1); the central conjecture that infinitely many of these solutions have p prime is equivalent to the existing open question of prime values among the odd-indexed Pell numbers, and the computational evidence of Table 2 is consistent with it. For $k \geq 3$, the classification is complete (Theorem 1): the only solution triple is $(239,13,4)$, so $k=1$ and $k=2$ are the only cases in which infinitude remains open. The classification framework established here consolidates the known results on this equation and positions the one genuinely open question, the $k=2$ primality question, within the existing literature on Pell numbers and the generalized Lebesgue–Ramanujan–Nagell family.

That a question as elementary to state as whether n^2+1 can be twice a prime power infinitely often should reduce, in its one remaining open case, to an unsolved question about prime values of a linear recurrence is a reminder of how narrow the boundary is between the arithmetic that can be settled by hand and the arithmetic that is still out of reach.

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