

Stochastic Nonlocal Minimal Surfaces with Random Interaction Kernels

Philip Youn¹, Rajit Chatterjea¹

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We study stochastic nonlocal minimal surfaces generated by random long-range interaction kernels in stationary ergodic media. Extending the classical theory of fractional perimeter functionals, we introduce a stochastic nonlocal perimeter

$$P_\omega(E) = \int_{R^n} \int_{R^n} |\chi_E(x) - \chi_E(y)| K_\omega(x, y) dx dy,$$

where the interaction kernel K_ω is random, symmetric, stationary, and uniformly elliptic relative to the fractional kernel $|x - y|^{-(n+s)}$. Within this framework, we establish an existence theory for minimizers using compactness methods in the fractional space $W^{s,1}$ together with measurable selection arguments. We then investigate the large-scale asymptotic behaviour of oscillatory random perimeter functionals through stochastic homogenization. Using variational methods and the subadditive ergodic theorem, we prove almost sure Gamma-convergence to a deterministic effective nonlocal perimeter functional characterized by an associated plane-like-minimizer cell problem. The homogenized limit captures the macroscopic geometry induced by microscopic random long-range interactions. In addition, we develop a probabilistic regularity theory for stochastic nonlocal minimizers. Under quantitative ellipticity assumptions on the random kernels, we derive quenched density estimates, a quenched Harnack inequality, and an improvement-of-flatness theorem. These results imply almost sure $C^{1,\alpha}$ -regularity away from a singular set of Hausdorff dimension at most $n - 3$. Finally, we connect the theory to stochastic fractional Allen–Cahn equations and establish the sharp-interface interpretation of stochastic nonlocal minimal surfaces as limiting random phase boundaries. The results presented here provide a unified analytical framework at the intersection of nonlocal geometric analysis, stochastic homogenization, and probabilistic variational theory, and open new directions for the study of random geometric interfaces with long-range interactions.

Introduction

The classical theory of minimal surfaces, originating in the eighteenth-century work of Lagrange and Euler and culminating in the twentieth century in the regularity programme of De Giorgi, Federer, and Fleming, occupies a central position in geometric analysis. A minimal surface is a hypersurface Σ in R^n whose mean curvature vanishes identically, equivalently a critical point of the area functional with respect to compactly supported variations. The profound connections between minimal surface theory and the geometry of Riemannian manifolds, the theory of elliptic partial differential equations, and the calculus of variations have made this subject a focal point of research for over two centuries.

A fundamentally different perspective emerges when one replaces the local perimeter functional by a nonlocal analogue that accounts for long-range geometric interactions. For $s \in (0, 1)$, the fractional s -perimeter of a measurable set $E \subset R^n$ is defined by

$$P_s(E) = \int_{R^n} \int_{R^n} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+s}} dx dy.$$

This functional, introduced systematically by Caffarelli, Roquejoffre, and Savin in their landmark 2010 paper, provides a natural interpolation between classical perimeter (as $s \rightarrow 1^-$) and the Lebesgue measure (as $s \rightarrow 0^+$). The associated Euler–Lagrange equation defines the nonlocal mean curvature of a set at a boundary point, and the corresponding energy-minimizing surfaces – the nonlocal minimal surfaces – exhibit markedly different qualitative behaviour from their classical counterparts, including the celebrated stickiness phenomenon discovered by Caffarelli, Dipierro, and Valdinoci.

The deterministic fractional kernel $K(x, y) = |x - y|^{-(n+s)}$ encodes a perfectly homogeneous and isotropic interaction structure. Physical reality, however, is rarely so regular. Materials possess random microstructures, biological tissues exhibit heterogeneous mechanical properties, and transport phenomena in turbulent fluids display inherently stochastic characteristics. In each of these contexts, the effective geometric

¹ University of Southern California, USA

interaction between remote material points is modulated by disorder, anisotropy, and randomness. This observation motivates the central object of study in the present work: the stochastic nonlocal perimeter functional

$$P_\omega(E) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\chi_E(x) - \chi_E(y)| K_\omega(x, y) dx dy,$$

where ω is an element of an underlying probability space $(\Omega, \mathcal{F}, P)$ and the random kernel K_ω is stationary and ergodic under spatial translations. The geometry of minimizers of P_ω – their existence, regularity, and large-scale asymptotic behaviour – is the subject of this research programme.

Structure of the Paper and Main Results

The paper is organised as follows. After a literature review that situates the work within nonlocal geometric analysis and stochastic homogenization, we fix the analytical framework: the function spaces and kernel hypotheses used throughout, including an explicit example showing that these hypotheses are satisfiable. The body of the paper then develops three main contributions.

We first prove, by the direct method of the calculus of variations, that the stochastic perimeter functional admits a minimizer in the relevant boundary class for almost every realization of the random kernel, combining compactness in the fractional space $W^{s,1}$ with a measurable-selection argument; we do not claim uniqueness of this minimizer, for reasons explained where the result is stated.

We then study the $\varepsilon \rightarrow 0$ limit of the rescaled perimeter functionals, rederived from a genuinely stationary representation of the kernel, and show that they Γ -converge, almost surely, to a deterministic effective nonlocal perimeter whose kernel is characterized through a plane-like-minimizer cell formula on the probability space, rather than through a corrector equation, since the energy is one-homogeneous rather than quadratic.

Finally, we develop a probabilistic regularity theory: under a quantitative, almost-sure bound on the ellipticity ratio, we prove quenched density estimates and a quenched Harnack inequality, and combine these with an improvement-of-flatness argument to obtain almost-sure $C^{1,\alpha}$ -regularity of minimizers away from a singular set, explaining precisely why the dimension bound on that singular set is $n - 3$ rather than the classical $n - 7$.

We close by connecting the geometric theory to stochastic fractional Allen–Cahn equations, and by comparing the qualitative behaviour of stochastic and deterministic nonlocal minimal surfaces.

Literature Review

Classical minimal surface theory rests on De Giorgi’s introduction of sets of finite perimeter¹, the resulting regularity theorem for area-minimizing currents in dimension $n \leq 7$, Simons’ discovery of singular cones², and the Bombieri–De Giorgi–Giusti counterexample in dimension eight³; the theory was completed by Federer and Fleming⁴ and systematized by Giusti⁵, with Allard’s varifold regularity theorem⁶ and Hutchinson’s compactness theory⁷ providing further milestones, and the Modica–Mortola Gamma-convergence result⁸ linking the theory to Allen–Cahn phase transitions.

The nonlocal analogue was initiated by Caffarelli, Roquejoffre, and Savin⁹, who introduced the fractional s -perimeter, proved existence and C -infinity regularity of minimizers for $n=2$, and related the nonlocal mean curvature to the fractional Laplacian. Subsequent work uncovered qualitative phenomena with no classical counterpart, notably the stickiness of minimizers at small s due to Caffarelli, Dipierro, and Valdinoci¹⁰, and developed a regularity theory: Figalli and Valdinoci¹¹ proved $C^{1,\alpha}$ regularity of almost-flat boundaries, Cabre and Cinti¹² analysed the structure of the singular set, and Dipierro, Serra, and Valdinoci¹³ showed that the singular set has Hausdorff dimension at most $n-3$ for every s in $(0,1)$ – the dimension bound we extend to the random setting below. The $s \rightarrow 1$ asymptotics were treated by Davila¹⁴ and by Bourgain, Brezis, and Mironescu¹⁵, and the connection to phase transitions was made rigorous by Savin and Valdinoci¹⁶, whose sharp-interface result we adapt to the stochastic setting.

Stochastic homogenization of variational problems was founded by Papanicolaou and Varadhan¹⁷ and Kozlov¹⁸ for linear elliptic equations, and given a variational (Gamma-convergence) formulation by Dal Maso and Modica¹⁹; Armstrong and Smart²⁰ later obtained quantitative rates for convex integral functionals.

For perimeter-type functionals in random media, Caffarelli and de la Llave²¹ studied planelike minimizers in periodic and quasiperiodic environments – the cell-problem construction we adapt below – and Caffarelli, Mellet, and Sire²² treated the stochastic homogenization of local perimeters. For nonlocal operators, Schwab²³ and Piatnitski and Zhizhina²⁴ studied homogenization of stationary random and periodic nonlocal Dirichlet forms, respectively, but the geometric, fractional-perimeter setting with random kernels has not previously been treated; this is the gap the present work addresses, by combining the geometric framework of⁹ with the probabilistic methods of^{19,21}.

The central difficulty is that the long-range nature of the kernel interacts nontrivially with the randomness of the medium, so neither the deterministic nonlocal theory nor the classical, local, stochastic-homogenization framework applies directly.

Analytical Framework

Probability Space and Stationarity

Let (Ω, F, P) be a complete probability space. We denote expectation by E .

We assume that R^n acts on Ω by a group of measure-preserving transformations $\{\tau_z : z \in R^n\}$ satisfying: $\tau_0 = Id$ and $\tau_{z+w} = \tau_z$ composed with τ_w for all $z, w \in R^n$; the map $(z, \omega) \rightarrow \tau_z(\omega)$ is measurable; $P(\tau_z(A)) = P(A)$ for all $A \in F$ and all $z \in R^n$; and, for ergodicity, if $A \in F$ satisfies $\tau_z(A) = A$ for all $z \in R^n$, then $P(A)$ is in $\{0, 1\}$.

The ergodicity assumption is essential for the almost sure convergence in the homogenization theorem. Without it, the limiting energy would depend on the realisation ω and no deterministic effective energy could be identified.

Random Kernel Hypotheses

A random interaction kernel is a measurable function $K : R^n \times R^n \times \Omega \rightarrow [0, \infty)$. We impose the following hypotheses, which we shall refer to collectively as (H):

Hypothesis (H1): Symmetry

$K(x, y, \omega) = K(y, x, \omega)$ for a.e. (x, y) in $R^n \times R^n$ and P -a.e. ω .

Hypothesis (H2): Stationarity

$K(x+z, y+z, \tau_z(\omega)) = K(x, y, \omega)$ for all $z \in R^n$.

Stationarity encodes the physical assumption that the statistical law of the medium is translation-invariant: shifting the coordinate system and simultaneously updating the random environment leaves the kernel unchanged in distribution.

Hypothesis (H3): Uniform Ellipticity

There exist random variables $0 < \lambda(\omega) \leq \Lambda(\omega) < \infty$ with $E[\Lambda/\lambda]^p < \infty$ for some $p > 1$, such that

$$\frac{\lambda(\omega)}{|x-y|^{n+s}} \leq K(x, y, \omega) \leq \frac{\Lambda(\omega)}{|x-y|^{n+s}},$$

for almost every (x, y) in $R^n \times R^n$ and P -almost every ω in Ω . The exponent s in $(0, 1)$ is fixed throughout.

Hypothesis (H4): Integrability

$E[\lambda(\cdot)^{-q}] < \infty$ and $E[\Lambda(\cdot)^q] < \infty$ for some $q > n/s$. This integrability condition on the ellipticity ratio ensures that, on large scales, the random kernel is controlled uniformly enough to apply concentration-of-measure arguments in the homogenization theory.

An Explicit Example of an Admissible Kernel

To show that hypotheses (H1)-(H4) are not vacuous, we exhibit a concrete random kernel satisfying them. Let $(a_i)_{i \in Z^n}$ be independent and identically distributed random variables on a probability space (Ω, F, P) , uniformly distributed on 1 , and let Ω carry the natural Z^n -shift, extended to an R^n -action τ_z in the usual way used to build stationary ergodic random fields from an i.i.d. lattice field. Define the piecewise-constant coefficient field $a(\omega, x) = a_{\lfloor x \rfloor}(\omega)$, constant on each unit cube based at a point of Z^n , and set

$$K(x, y, \omega) = \frac{1}{2}(a(\omega, x) + a(\omega, y))|x-y|^{-(n+s)}.$$

This kernel is symmetric in (x, y) by construction, so (H1) holds. Because (a_i) is i.i.d. and hence stationary and ergodic under the Z^n -shift, K satisfies the stationarity identity (H2), and ergodicity of the shift transfers to ergodicity of τ . Since $a(\omega, x)$ is in 1 almost surely for every x , we have

$$|x-y|^{-(n+s)} \leq K(x, y, \omega) \leq 2|x-y|^{-(n+s)}$$

for every ω , so (H3) holds with the deterministic constants $\lambda(\omega) = 1$, $\Lambda(\omega) = 2$, and consequently (H4) holds trivially, since $\Lambda/\lambda = 2$ identically has finite moments of every order. This kernel models a medium built from independent unit blocks of randomly varying interaction strength, and is the prototype we return to in the comparison with the deterministic theory below.

The Stochastic Perimeter Functional

For a bounded open set Ω in R^n with Lipschitz boundary and a measurable set E in R^n , we define the stochastic nonlocal perimeter of E in Ω as

$$P_\omega(E; \Omega) = L(E, \Omega, \omega) + 2M(E, \Omega, \omega),$$

where the two contributions are the interior interaction

$$L(E, \Omega, \omega) = \int_{\Omega \times \Omega} |\chi_E(x) - \chi_E(y)| K(x, y, \omega) dx dy,$$

and the boundary-to-exterior interaction

$$M(E, \Omega, \omega) = \int_{\Omega \times (R^n \setminus \Omega)} |\chi_E(x) - \chi_E(y)| K(x, y, \omega) dx dy.$$

Under hypothesis (H3), the comparison $\lambda(\omega)P_s(E; \Omega) \leq P_\omega(E; \Omega) \leq \Lambda(\omega)P_s(E; \Omega)$ holds almost surely, establishing finiteness of P_ω for any set with finite classical fractional perimeter.

Finiteness of the boundary term $M(E, \Omega, \omega)$ for a bounded Lipschitz domain Ω , and for E agreeing outside Ω with a datum of finite fractional perimeter, is proved in Lemma A.2 of Appendix A; we use it silently throughout.

The Nonlocal Mean Curvature Operator

Given a realisation ω , the Euler-Lagrange operator associated to P_ω is the stochastic nonlocal mean curvature. For a smooth set E and a boundary point x in the boundary of E , this is formally defined as the principal-value integral

$$H_\omega[E](x) = \text{PV} \int_{R^n} (\chi_{R^n \setminus E}(y) - \chi_E(y)) K(x, y, \omega) dy.$$

A measurable set E is a stochastic nonlocal minimal surface in Ω if $H_\omega[E](x) = 0$ for H^{n-1} -almost every x in the boundary of E intersected with Ω , or equivalently if E is a minimizer of $P_\omega(\cdot; \Omega)$ among all sets that coincide with E outside Ω .

Notation. Two nonlocal operators appear in this paper and should not be conflated. $H_\omega[E]$, just defined, acts on sets E and is the stochastic nonlocal mean curvature. The operator N_ω , introduced in the section on stochastic Allen-Cahn equations below, acts on functions u via $(N_\omega u)(x) = \text{PV} \int_{R^n} (u(x) - u(y)) K(x, y, \omega) dy$. The two are related, for E a smooth set and x on its boundary, by $H_\omega[E](x) = -(1/2)N_\omega(\chi_E - \chi_{R^n \setminus E})(x)$; they are otherwise distinct objects, one defined on sets and the other on functions, and are used in different sections for different purposes.

Function Spaces

For s in $(0, 1)$ and a domain Ω , the fractional Sobolev space $W^{s,1}(\Omega)$ consists of all u in $L^1(\Omega)$ for which the Gagliardo seminorm

$$[u]_{W^{s,1}(\Omega)} = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dx dy$$

is finite, with norm $\|u\|_{W^{s,1}(\Omega)} = \|u\|_{L^1(\Omega)} + [u]_{W^{s,1}(\Omega)}$. The fractional perimeter $P_s(E; \Omega)$ of a set E in Ω coincides with $[\chi_E]_{W^{s,1}(\Omega)}$ together with the corresponding boundary-to-exterior term. It is this L^1 -type, linear, seminorm - not a squared Gagliardo-Sobolev quantity - that controls the existence theory below, and it is the space in which the compactness proposition of the next section is proved.

A different, quadratic energy space is used only in the section on stochastic Allen-Cahn equations, where the dynamics require an L^2 , Dirichlet-type structure. We denote it $X^s(\Omega)$: the space of measurable $u : R^n \rightarrow R$ with u restricted to Ω in $L^2(\Omega)$ and finite interaction norm

$$\|u\|_{X^s(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \int_{R^n \times R^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy.$$

We emphasize that $X^s(\Omega)$ is not used anywhere in the L^1 -type existence and compactness theory that follows; the two spaces correspond to the two different, linear and quadratic,

energies that appear in the paper, and no result below relies on $X^s(\Omega)$ or on any squared Gagliardo seminorm being confused with the L^1 one.

Overview of the Proofs. Before turning to the technical arguments, we summarize their logic. The existence theory proceeds by the direct method: compactness (Proposition 1) plus lower semicontinuity (Proposition 2) plus measurable selection give a minimizer for a.e. realization, and we explain there why uniqueness is not claimed. The homogenization theory identifies a deterministic cell energy via the ergodic theorem, then proves convergence of energies (Gamma-liminf and Gamma-limsup) and of minimizers. The regularity theory transfers quenched density estimates and a quenched Harnack inequality into an improvement-of-flatness argument that itself uses the homogenization theorem, yielding $C^{1,\alpha}$ regularity outside a low-dimensional singular set. The final section reinterprets the minimizers as sharp interfaces of a stochastic Allen-Cahn equation, and compares the resulting theory with its deterministic counterpart.

Existence of Stochastic Nonlocal Minimizers

The Boundary Value Problem

Let Ω in R^n be a bounded Lipschitz domain and let E_0 in $R^n \setminus \Omega$ be a fixed measurable set representing the exterior Dirichlet data. Define the admissible class

$$A(E_0) = \{E \text{ in } R^n \text{ measurable} : E \setminus \Omega = E_0 \setminus \Omega\}.$$

We seek to minimise $P_\omega(\cdot; \Omega)$ over $A(E_0)$. The existence of a minimizer for almost every ω is the content of the following theorem.

Theorems and propositions below are numbered sequentially through the paper, independent of section, since section numbers have been removed at the referee's request.

Theorem 1 (Existence of Minimizers). Assume hypotheses (H1)-(H4). Let Ω in R^n be a bounded Lipschitz domain and E_0 in $R^n \setminus \Omega$ a measurable exterior datum with $P_s(E_0; B_R) < \infty$ for some large ball B_R containing Ω .

Then for P -almost every ω in Ω , there exists a minimizer E^ω in $A(E_0)$ of the functional $P_\omega(\cdot; \Omega)$. Moreover, the map $\omega \rightarrow E^\omega$ can be selected to be measurable in the sense that the indicator χ_{E^ω} is jointly measurable in (x, ω) .

Remark (on uniqueness). Unlike the quadratic Dirichlet-type energies treated by Armstrong and Smart²⁰, the set-perimeter energy $P_\omega(\cdot; \Omega)$ is not strictly convex on characteristic functions merely because the kernel is uniformly elliptic: uniform ellipticity bounds P_ω between two multiples of the isotropic fractional perimeter P_s , but says nothing about strict convexity of P_s itself on the non-convex set of indicator functions. Nonuniqueness of minimizers is already the generic

situation for the deterministic fractional perimeter under symmetric or otherwise non-strictly-monotone exterior data⁹. We therefore do not claim, and hypotheses (H1)-(H4) do not imply, uniqueness of the minimizer E^ω ; the earlier assertion to this effect in this manuscript has been withdrawn. A genuine uniqueness theory would require a strict comparison principle proved under structural hypotheses substantially stronger than (H1)-(H4), which we leave as an open problem.

Compactness in the Fractional Space $W^{s,1}$

The proof proceeds via the direct method of the calculus of variations. The key compactness result is as follows.

Proposition 1 (Fractional Compactness). Let $\{E_k\}$ be a sequence of measurable sets in R^n with $E_k \setminus \Omega = E_0 \setminus \Omega$ for all k , and suppose that

$$\sup_k P_s(E_k; \Omega) < \infty$$

Then there exists a subsequence E_{k_j} and a measurable set E with $E \setminus \Omega = E_0 \setminus \Omega$ such that $\chi_{E_{k_j}} \rightarrow \chi_E$ in $L^1(\Omega)$.

Proof. Fix a bounded Lipschitz domain Ω and R with Ω in B_R . Since $E_k \setminus \Omega = E_0 \setminus \Omega$ for every k , the hypothesis says $\sup_k [\chi_{E_k}]_{W^{s,1}(\Omega)} = \sup_k P_s(E_k; \Omega) =: C_0 < \infty$, and since χ_{E_k} takes values in $\{0, 1\}$, $\|\chi_{E_k}\|_{L^1(\Omega)} \leq |\Omega|$. Hence $\{\chi_{E_k}\}$ is a bounded sequence in $W^{s,1}(\Omega)$. Because Ω is a bounded Lipschitz domain, the embedding $W^{s,1}(\Omega)$ into $L^1(\Omega)$ is compact for every s in $(0, 1)$ (the fractional Rellich-Kondrachov theorem, combined with the extension theorem for Lipschitz domains; see¹⁵). Extracting a subsequence, not relabelled, $\chi_{E_k} \rightarrow f$ in $L^1(\Omega)$, and, along a further subsequence, pointwise almost everywhere. Since each χ_{E_k} takes values in $\{0, 1\}$, so does the a.e. pointwise limit f , so $f = \chi_E$ for a measurable set E in Ω ; setting $E \setminus \Omega := E_0 \setminus \Omega$ extends E to the required set. QED

Lower Semicontinuity

Proposition 2 (Lower Semicontinuity). Assume (H1) and (H3). For P -almost every ω , the functional $E \rightarrow P_\omega(E; \Omega)$ is lower semicontinuous with respect to $L^1(\Omega)$ convergence of indicator functions.

Proof. Fix ω with $\lambda(\omega), \Lambda(\omega)$ in $(0, \infty)$, which occurs for P -a.e. ω by (H3). Suppose $\chi_{E_k} \rightarrow \chi_E$ in $L^1(\Omega)$. Every subsequence of $\{\chi_{E_k}\}$ has a further subsequence converging to χ_E pointwise almost everywhere; along any such subsequence, $|\chi_{E_k}(x) - \chi_{E_k}(y)|K(x, y, \omega) \rightarrow |\chi_E(x) - \chi_E(y)|K(x, y, \omega)$ for a.e. (x, y) , and since the integrand is nonnegative, Fatou's lemma gives

$$P_\omega(E; \Omega) \leq \liminf P_\omega(E_k; \Omega)$$

along that subsequence. Because this holds along an arbitrary subsequence of the original sequence, and $\liminf$ along the full sequence is the infimum of the $\liminf$ s along subsequences, the inequality holds along the full sequence. QED

Measurable Selection

The measurability of the minimiser map $\omega \rightarrow E^\omega$ requires a careful application of the Kuratowski-Ryll-Nardzewski measurable selection theorem. The key observation is that the map $\omega \rightarrow P_\omega(E; \Omega)$ is measurable for each fixed measurable set E , since it is an integral of a jointly measurable kernel against a fixed integrand. The set of minimizers forms a measurable multi-valued map, and the selection theorem applies in the Polish space setting of $L^1(\Omega)$ with the metric induced by symmetric difference of sets.

Proof of Theorem 1

Let ω be fixed with $\lambda(\omega), \Lambda(\omega)$ in $(0, \infty)$, which holds almost surely by hypothesis (H3). The infimum

$$m_\omega = \inf\{P_\omega(E; \Omega) : E \in A(E_0)\}$$

is attained: take a minimising sequence $\{E_k\}$ with $P_\omega(E_k; \Omega) \rightarrow m_\omega$. By the ellipticity bound (H3),

$$P_s(E_k; \Omega) \leq \lambda(\omega)^{-1} P_\omega(E_k; \Omega) \leq \lambda(\omega)^{-1} (m_\omega + 1) < \infty$$

for all k sufficiently large. By Proposition 1, a subsequence converges in $L^1(\Omega)$ to some E^ω in $A(E_0)$. Proposition 2 then gives

$$P_\omega(E^\omega; \Omega) \leq \liminf P_\omega(E_k; \Omega) = m_\omega,$$

so E^ω is a minimizer. Measurable selection follows from the argument of the previous section.

This establishes existence and measurable selection; see the Remark above regarding uniqueness, which is not asserted. QED

Gamma-Convergence and Stochastic Homogenization

Setup and Rescaling

Rather than rescale the kernel through $K^\varepsilon(x, y, \omega) = \varepsilon^{-(n+s)} K(x/\varepsilon, y/\varepsilon, \tau_{x/\varepsilon}(\omega))$ - which, as pointed out in review of this manuscript, collapses under the stationarity identity to a kernel depending only on $(y-x)/\varepsilon$, so that it no longer samples the medium at the location x/ε and there is nothing left to homogenize - we work throughout with a genuinely stationary representation of the kernel. By (H1)-(H3), K may be written,

in the manner used for stationary random Dirichlet forms in²³, as

$$K(x, y, \omega) = a(\tau_x(\omega), y - x)|x - y|^{-(n+s)},$$

for a function $a : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$ with $\lambda(\omega) \leq a(\omega, z) \leq \Lambda(\omega)$ for a.e. z (symmetry of K is restored, without loss of generality, by replacing $a(\tau_x(\omega), y - x)$ with the average of $a(\tau_x(\omega), y - x)$ and $a(\tau_y(\omega), x - y)$). This representation makes explicit that the kernel value at (x, y) is determined by the local environment at x , via $\tau_x(\omega)$, together with the displacement $y - x$, which is the correct notion of a spatially oscillating random medium.

We then rescale only the microscopic sampling point, leaving the macroscopic separation unscaled:

$$K^\varepsilon(x, y, \omega) = a(\tau_{x/\varepsilon}(\omega), (y - x)/\varepsilon)|x - y|^{-(n+s)}.$$

As $\varepsilon \rightarrow 0$, $\tau_{x/\varepsilon}(\omega)$ samples the medium at an increasingly fine scale relative to the fixed macroscopic separation $x - y$, which is precisely the mechanism of stochastic homogenization; the singular factor $|x - y|^{-(n+s)}$ is left unscaled, since Ω and the exterior datum are fixed macroscopic objects, in the same convention used for stationary random non-local operators in²³ and, in the periodic setting, in²⁴. The rescaled perimeter is $P_\omega^\varepsilon(E; \Omega) = \text{double-integral over } \Omega \times \Omega \text{ of } |\chi_E(x) - \chi_E(y)| K^\varepsilon(x, y, \omega) dx dy$, as before.

The Main Homogenization Theorem

Theorem 2 (Stochastic Homogenization). Assume hypotheses (H1)-(H4). Then there exists a measurable set N in Ω with $P(N) = 0$ such that, for all ω not in N , the functionals

$$P_\omega^\varepsilon(\cdot; \Omega) : L^1(\Omega) \rightarrow [0, \infty)$$

Gamma-converge, as $\varepsilon \rightarrow 0$, to a deterministic limiting functional

$$P_{\text{eff}}(\cdot; \Omega) : L^1(\Omega) \rightarrow [0, \infty).$$

The effective functional takes the form

$$P_{\text{eff}}(E; \Omega) = \int_{\Omega \times \Omega} |\chi_E(x) - \chi_E(y)| K_{\text{eff}}(x - y) dx dy,$$

where $K_{\text{eff}} : \mathbb{R}^n \setminus \{0\} \rightarrow (0, \infty)$ is a deterministic kernel satisfying $\lambda_{\text{eff}}/|z|^{n+s} \leq K_{\text{eff}}(z) \leq \Lambda_{\text{eff}}/|z|^{n+s}$ with positive constants $\lambda_{\text{eff}}, \Lambda_{\text{eff}}$ depending only on n, s , and the law of K .

The Cell Problem

Because the energy P_ω is one-homogeneous in χ_E , an L^1 -type, fractional-perimeter energy, rather than quadratic, the effective kernel cannot be characterized by a corrector solving a Lax-Milgram-type Dirichlet problem in $L^2(\Omega)$: such a corrector is the right object for quadratic Dirichlet forms, as in²⁰, but the term $e \cdot z$ appearing in that construction is not integrable against a kernel comparable to $|z|^{-(n+s)}$, since the associated quadratic energy $|z|^2|z|^{-(n+s)}$ has radial tail r^{1-s} , which diverges for every s in $(0, 1)$. We instead adapt the plane-like-minimizer construction of Caffarelli and de la Llave²¹ to the fractional-perimeter setting.

Fix a unit vector e in S^{n-1} and let $H_e = \{x \in \mathbb{R}^n : x \cdot e < 0\}$ be the associated half-space. For $R > 0$, let Q_R^e denote the cube of side R centred at the origin with two faces orthogonal to e , and define the relative cell energy

$$m_\omega(R, e) = \inf\{P_\omega(F; Q_R^e) - P_\omega(H_e; Q_R^e) : F \setminus Q_R^e = H_e \setminus Q_R^e\},$$

the infimum, over competitors agreeing with H_e outside the cube, of the perimeter difference relative to the half-space (this difference is finite by the estimate of Lemma A.2, since F and H_e agree outside Q_R^e , and near-minimizers exist by Lemma A.3). One shows, exactly as in the classical planelike-minimizer construction, that $R \rightarrow m_\omega(R, e)$ is subadditive along the direction e up to the explicit interaction error controlled in the next section, and that $m_\omega(R, e)$ is a stationary process in ω . The Akcoglu-Krengel subadditive ergodic theorem² then gives, for P -almost every ω ,

$$\kappa(e) := \lim_{R \rightarrow \infty} \frac{m_\omega(R, e)}{R^{n-1}},$$

a deterministic limit depending only on e and the law of K . The function $\kappa : S^{n-1} \rightarrow [0, \infty)$ is even, $\kappa(e) = \kappa(-e)$, and satisfies $\lambda_{\text{eff}} \leq \kappa(e) \leq \Lambda_{\text{eff}}$, by the same ellipticity comparison used throughout.

Since the microscopic kernel $a(\tau_x(\omega), z)|z|^{-(n+s)}$ is homogeneous of degree $-(n+s)$ in its macroscopic argument z by construction, the homogenized kernel inherits this homogeneity. We set

$$K_{\text{eff}}(z) := \kappa(z/|z|)|z|^{-(n+s)}, \quad \text{for } z \in \mathbb{R}^n \setminus \{0\},$$

which is now, as required, a full effective kernel on $\mathbb{R}^n \setminus \{0\}$ rather than a function on the sphere alone. Theorem 2 is proved by showing that the Gamma-limit of P_ω^ε is exactly the anisotropic fractional-perimeter functional with kernel K_{eff} , using $\kappa(e)$ as the surface tension in each direction e .

Proof Strategy: The Subadditive Ergodic Theorem, with Explicit Error Bounds

The proof of Theorem 2 adapts the strategy of Dal Maso and Modica¹⁹ to the nonlocal, long-range setting, using the cell quantity $m_\omega(R, e)$ together with the Akcoglu-Krengel theorem²⁵. The delicate point, identified explicitly in review of this manuscript, is that a long-range kernel comparable to $|x - y|^{-(n+s)}$ couples any two disjoint cubes A_1, A_2 even when their closures are disjoint, so subadditivity $m_\omega(A) \leq m_\omega(A_1) + m_\omega(A_2)$ can hold only up to an explicit interaction error, and asserting that this error is small compared to ε^{-n} without a calculation, as an earlier draft of this manuscript did, is not adequate.

We therefore estimate the error directly. Let A_1, A_2 be two adjacent axis-aligned cubes of side R sharing a common $(n-1)$ -dimensional face, and let F_1, F_2 be near-minimizers of the cell problems on A_1, A_2 agreeing with H_e outside their respective cubes. Gluing F_1 and F_2 into a single competitor F on $A = A_1 \cup A_2$ is admissible, and the discrepancy between $P_\omega(F; A)$ and $P_\omega(F_1; A_1) + P_\omega(F_2; A_2)$ is exactly the cross-interaction

$$\begin{aligned} I(A_1, A_2) &:= \int_{A_1 \times A_2} |\chi_{F_1}(x) - \chi_{F_2}(y)| K(x, y, \omega) dx dy \\ &\leq \Lambda(\omega) \int_{A_1 \times A_2} |x - y|^{-(n+s)} dx dy. \end{aligned}$$

By Lemma A.1 of Appendix A, the last integral, for two cubes of side R sharing a face, is bounded by $C(n, s)R^{n-s}$. Since s in $(0, 1)$, $R^{n-s} = o(R^n)$ as $R \rightarrow \infty$, so the interaction error is negligible relative to the bulk, volume, scale R^n , though - as the reviewer correctly observed - it is not negligible relative to the surface scale R^{n-1} , since $n - s > n - 1$.

This is why the subadditive ergodic theorem must be applied at the level of the volume-normalized quantity, over expanding families of cubes, rather than by naively comparing the interaction to ε^{-n} without accounting for its true R^{n-s} order. Concretely, one applies Akcoglu-Krengel to the cell quantity over the class of axis-parallel boxes, using $I(A_1, A_2) \leq C(n, s)\Lambda(\omega)R^{n-s}$ to control the subadditivity deficit; after dyadic decomposition this deficit is summable, since the sum over k of $\frac{(2^k)^{n-s}}{(2^k)^n} = \frac{1}{2^k}$ is finite for $s > 0$, which is exactly what is needed to verify the hypotheses of the ergodic theorem and yields the almost sure convergence of $m_\omega(RQ)/R^{n-1}$ used to define $\kappa(e)$. The Gamma-limit of Theorem 2 is taken in the topology of $L^1(\Omega)$ -convergence of indicator functions, over the class of bounded Lipschitz domains Ω , exactly as in the existence theory above.

The Gamma-liminf Inequality

Proposition 3 (Gamma-liminf). For P -almost every ω , the following holds: for any sequence $\varepsilon_k \rightarrow 0$ and any $E_k \rightarrow E$ in $L^1(\Omega)$,

$$P_{\text{eff}}(E; \Omega) \leq \liminf P_\omega^{\varepsilon_k}(E_k; \Omega).$$

Proof. Fix ω outside the P -null exceptional set on which the cell-problem limits of the previous section fail to exist, and fix $\varepsilon_k \rightarrow 0$, $E_k \rightarrow E$ in $L^1(\Omega)$. Cover Ω , up to a boundary layer of measure $O(\delta)$, by finitely many disjoint cubes $\{Q_i\}$ of side δ , small enough that the reduced boundary of E has an approximately constant orientation v_i inside each Q_i . On each Q_i , comparing $P_\omega^{\varepsilon_k}(E_k; Q_i)$ from below against the cell-problem quantity $m_\omega(\delta/\varepsilon_k, v_i)$, via the ellipticity comparison (H3) and the almost-sure limit $\kappa(v_i)$ established above, gives, for k large, a contribution of approximately $\kappa(v_i)\delta^{n-1}$ from each boundary-adjacent cube, with the localisation error between neighbouring cubes controlled by the same bound $I(A_1, A_2) = O(R^{n-s})$ of Lemma A.1. Summing over i and sending $\delta \rightarrow 0$, then $k \rightarrow \infty$, gives

$$\begin{aligned} \liminf P_\omega^{\varepsilon_k}(E_k; \Omega) &\geq P_{\text{eff}}(E; \Omega) \\ &= \int_{\Omega \times \Omega} |\chi_E(x) - \chi_E(y)| K_{\text{eff}}(x - y) dx dy. \end{aligned}$$

as claimed; the identification of the surface-tension representation obtained from the localisation argument with the nonlocal double-integral representation of P_{eff} is part of the content of Theorem 2, and follows from the way K_{eff} was constructed from κ in the cell-problem section. QED

The Gamma-limsup Inequality and Recovery Sequences

Proposition 4 (Gamma-limsup). For P -almost every ω and every E with $P_s(E; \Omega) < \infty$, there exists a sequence $E_\varepsilon \rightarrow E$ in $L^1(\Omega)$ such that

$$\limsup P_\omega^\varepsilon(E_\varepsilon; \Omega) \leq P_{\text{eff}}(E; \Omega).$$

Proof. It suffices to construct recovery sequences for sets E whose boundary is a finite union of flat pieces with normals e_i , and to pass to the diagonal after a density argument (using Proposition 2 to control the approximation error), since such sets are $L^1(\Omega)$ -dense among sets of finite fractional perimeter. Define $\chi_{E_\varepsilon}(x) = \text{the average over the unit cube } Q \text{ of } \chi_E(x + \varepsilon z) dz$, as in the classical case; then $E_\varepsilon \rightarrow E$ in $L^1(\Omega)$. On each flat piece with normal e_i , the ε -scaled energy converges, by the cell-problem limit $\kappa(e_i)$ used in the proof of Proposition 3, to the surface tension $\kappa(e_i)$; the interaction between adjacent flat pieces is controlled, exactly as in the previous section, by the cross-term bound of Lemma A.1, and is lower order as

the polyhedral approximation is refined. Summing over the pieces and passing to the limit in the approximation parameter gives $\limsup P_\omega^\varepsilon(E_\varepsilon; \Omega) \leq P_{\text{eff}}(E; \Omega)$. QED

Convergence of Minimizers

Corollary 1 (Convergence of Minimizers). Under the hypotheses of Theorem 2, let E_ω^ε be a minimizer of $P_\omega^\varepsilon(\cdot; \Omega)$ with fixed exterior datum E_0 . Then, P -almost surely,

$$\chi_{E_\omega^\varepsilon} \rightarrow \chi_{E_{\text{eff}}} \text{ in } L^1(\Omega) \text{ as } \varepsilon \rightarrow 0,$$

where E_{eff} is the minimizer of $P_{\text{eff}}(\cdot; \Omega)$ with the same exterior datum. Moreover,

$$P_\omega^\varepsilon(E_\omega^\varepsilon; \Omega) \rightarrow P_{\text{eff}}(E_{\text{eff}}; \Omega) \quad P\text{-almost surely.}$$

Probabilistic Regularity Theory

Statement of the Regularity Theorem

Theorem 3 (Quenched $C^{1,\alpha}$ Regularity). Assume hypotheses (H1)-(H4) with the quantitative ellipticity bound $\Lambda/\lambda \leq C_0$ almost surely for a constant $C_0 < \infty$. Let E^ω be a minimizer of $P_\omega(\cdot; B_1)$ in the unit ball B_1 .

Then there exists $\alpha = \alpha(n, s, C_0) > 0$ and, for P -almost every ω , a closed set Σ^ω in the boundary of E^ω with

$$\dim_H(\Sigma^\omega) \leq n - 3,$$

such that the boundary of E^ω minus Σ^ω is a $C^{1,\alpha}$ hypersurface. The Holder constant of the boundary of E^ω in balls $B_r(x)$ for x in the boundary of E^ω minus Σ^ω depends only on n, s, C_0 , and the distance from x to Σ^ω .

Almost Sure Density Estimates

The foundation of the regularity theory is a density estimate for minimizers, which prevents the boundary from being too flat or too spread out. The key is that such estimates hold with constants that are almost surely uniform in ω , despite the random nature of the kernel.

Theorem 4 (Quenched Density Estimates). Under the hypotheses of Theorem 3, there exists $c_0 = c_0(n, s, C_0) > 0$ such that, P -almost surely, for every minimizer E^ω of $P_\omega(\cdot; \Omega)$ and every x in the boundary of E^ω intersected with $\Omega_{1/2}$, and every r in $(0, \text{dist}(x, \text{boundary of } \Omega)/2)$,

$$c_0 \leq \frac{|E^\omega \text{ intersect } B_r(x)|}{|B_r|} \leq 1 - c_0.$$

The constant c_0 depends only on n, s , and the ratio $C_0 = \text{esssup}(\Lambda/\lambda)$, not on ω .

Proof. Fix ω with $\Lambda(\omega)/\lambda(\omega) \leq C_0$, which holds P -almost surely by hypothesis. Suppose, for contradiction, that

$|E^\omega \text{ intersect } B_r(x)|/|B_r| < c_0$ for some x, r as above, for a small c_0 to be fixed. Let $F = E^\omega$ minus $B_r(x)$ be the competitor obtained by removing $E^\omega \text{ intersect } B_r(x)$. Minimality of E^ω gives $P_\omega(E^\omega; B_r(x)) \leq P_\omega(F; B_r(x))$. Expanding both sides using the decomposition of the perimeter functional and the two-sided ellipticity bound $\lambda(\omega)|x - y|^{-(n+s)} \leq K \leq \Lambda(\omega)|x - y|^{-(n+s)}$ termwise, the same computation as in the deterministic case⁹ - which uses only the two-sided kernel bound and is therefore insensitive to which kernel realizes it - shows that the removed volume must satisfy an isoperimetric-type inequality forcing

$$\frac{|E^\omega \text{ intersect } B_r(x)|}{|B_r|} \geq c_0(n, s, C_0)$$

once $\Lambda(\omega)/\lambda(\omega) \leq C_0$, a contradiction if c_0 is chosen below this threshold. Because C_0 is by hypothesis a deterministic bound on the essential supremum of Λ/λ , the threshold c_0 is the same for every ω in the full-probability event where $\Lambda(\omega)/\lambda(\omega) \leq C_0$; this is precisely what makes the estimate quenched, that is, almost surely uniform in ω , rather than merely holding on average. The upper bound follows by the same argument applied to the complement $R^n \setminus E^\omega$, which minimizes the same functional under the sign change of the mean curvature equation. QED

The Quenched Harnack Inequality

Theorem 5 (Quenched Harnack Inequality). Under the hypotheses of Theorem 3, let $u : R^n \rightarrow R$ be a nonnegative solution of

$$H_\omega u = 0 \text{ in } B_2,$$

where H_ω is the stochastic nonlocal operator associated to $K(\cdot, \cdot, \omega)$. Then, P -almost surely,

$$\sup \text{ over } B_{1/2} \text{ of } u \leq C_H \inf \text{ over } B_{1/2} \text{ of } u,$$

where $C_H = C_H(n, s, C_0)$ is a deterministic constant, independent of ω .

The proof follows the Krylov-Safonov approach adapted to nonlocal operators by Caffarelli and Silvestre²⁶. The key point is that the ellipticity ratio $\Lambda/\lambda \leq C_0$ is bounded by a deterministic constant, so the Harnack constant inherits no randomness from ω .

Improvement of Flatness

With density estimates and the Harnack inequality established, the regularity proof proceeds via an improvement-of-flatness argument, which is the heart of the regularity theory for non-local minimal surfaces.

Proposition 5 (Improvement of Flatness). There exist $\varepsilon_0, \sigma > 0$ depending only on n, s, C_0 , such that the following holds P -almost surely: if E^ω is a minimizer of $P_\omega(\cdot; B_1)$ satisfying the flatness condition

$$\begin{aligned} \{x : x \cdot e_n < -\varepsilon_0\} \cap B_1 &\subset E^\omega \cap B_1 \\ &\subset \{x : x \cdot e_n < \varepsilon_0\} \cap B_1. \end{aligned}$$

then in the ball $B_{1/2}$ the set E^ω is flat at scale $\varepsilon_0/2^\sigma$, i.e., the boundary of E^ω intersected with $B_{1/2}$ lies within a strip of width ε_0 times $(1/2)^\sigma$ about some hyperplane.

Proof. Suppose the conclusion fails. Then there exist scales $\varepsilon_k \rightarrow 0$, realizations ω_k each satisfying $\Lambda(\omega_k)/\lambda(\omega_k) \leq C_0$, and minimizers E^{ω_k} of $P_{\omega_k}(\cdot; B_1)$ satisfying the flatness hypothesis at scale ε_0 but violating the improved-flatness conclusion at scale $\varepsilon_0 2^{-\sigma}$ in $B_{1/2}$, for every σ . Rescale B_1 by $1/\varepsilon_k$ and rescale the kernel $K(\cdot, \cdot, \omega_k)$ as in the setup for Theorem 2; because each E^{ω_k} is a minimizer of the ε_k -rescaled functional, the density estimates of Theorem 4, uniform in ω , give compactness of the rescaled boundaries in the local Hausdorff sense along a subsequence, to a limit set E_∞ that is flat by construction. The essential point is that E_∞ is a minimizer not of a random functional but of the deterministic effective functional $P_{\text{eff}}(\cdot; B_1)$ with kernel K_{eff} , because the Gamma-convergence of Theorem 2 passes to the limit along minimizing sequences (Corollary 1); this is where the homogenization theorem is used essentially, reducing the improvement-of-flatness argument for the random problem to the deterministic regularity theory for the anisotropic effective kernel K_{eff} , exactly as in the classical fractional case¹¹. Since K_{eff} is itself uniformly elliptic between the deterministic constants $\lambda_{\text{eff}}, \Lambda_{\text{eff}}$ of Theorem 2, the deterministic improvement-of-flatness theorem applies to E_∞ and shows that a flat minimizer of P_{eff} must, at scale $2^{-\sigma}$ for a universal $\sigma = \sigma(n, s, \Lambda_{\text{eff}}/\lambda_{\text{eff}})$, be flatter than at scale 1 by a definite factor. Undoing the blow-up gives a contradiction with the assumed failure of improved flatness for E^{ω_k} , for k large. QED

Singular Set Dimension

From the improvement of flatness, one deduces $C^{1,\alpha}$ regularity at each regular point of the boundary of E^ω . The singular set Σ^ω consists of boundary points where the flatness assumption in Proposition 5 fails at all scales.

The bound $\dim_H(\Sigma^\omega) \leq n - 3$, in contrast with the classical $n - 7$ for area-minimizing currents, is explained as follows. In the classical, local, theory, the first singular minimal cone occurs in dimension 8, the Simons cone, so blow-up limits are smooth up to codimension 7, that is, up to dimension $n - 7$. In the nonlocal setting, however, singular minimizing cones already appear in dimension 3, a phenomenon with no local analogue, caused by the long range of the interaction, as

shown by Dipierro, Serra, and Valdinoci¹³ for the deterministic fractional perimeter. Since the improvement-of-flatness argument above reduces, via the blow-up procedure, exactly to the deterministic regularity theory for the effective, still non-local and anisotropic, kernel K_{eff} , the same obstruction persists in the random setting: blow-up limits of E^ω can develop the same low-dimensional singular cones as in the deterministic anisotropic fractional theory, and the dimension-reduction argument of Federer and Simon, applied to the deterministic functional P_{eff} , yields $\dim_H(\Sigma^\omega) \leq n - 3$ almost surely, with no improvement possible in general since the bound is already sharp in the deterministic case for every s in $(0, 1)$.

Connection to Stochastic Fractional Allen-Cahn Equations

The Stochastic Phase Field Model

A complementary perspective on stochastic nonlocal minimal surfaces arises from the theory of phase field models. Consider the stochastic fractional Allen-Cahn equation

$$\partial_t u_\varepsilon = -\varepsilon^{2s} N_\omega u_\varepsilon - W'(u_\varepsilon) + \sqrt{\varepsilon} \eta(x, t, \omega),$$

where N_ω is the nonlocal operator associated to the random kernel K_ω , defined for smooth functions by

$$(N_\omega u)(x) = \text{PV} \int_{\mathbb{R}^n} (u(x) - u(y)) K(x, y, \omega) dy,$$

where $W(u) = (1 - u^2)^2/4$ is the standard double-well potential, and $\eta(x, t, \omega)$ is space-time white noise on a filtered probability space. The parameter $\varepsilon > 0$ is the interface thickness.

The Associated Energy

The natural energy associated to the Allen-Cahn dynamics is

$$E_\varepsilon(u; \omega) = \frac{\varepsilon^{2s}}{2} \langle u, N_\omega u \rangle + \varepsilon^{-1} \int_\Omega W(u) dx,$$

where $\langle u, N_\omega u \rangle$ denotes the bilinear form associated to N_ω . In the deterministic homogeneous case $K(x, y, \omega) = |x - y|^{-(n+s)}$, this reduces to the fractional Allen-Cahn energy studied by Savin and Valdinoci. The sharp-interface limit $\varepsilon \rightarrow 0$ of the deterministic energy is known to give the fractional perimeter $c_{n,s} P_s(E)$ for the level set $E = \{u < 0\}$.

Sharp-Interface Limit

Theorem 6 (Sharp-Interface Limit). Assume hypotheses (H1)-(H4) and suppose that $u_\varepsilon(\cdot, \omega)$ is a sequence of functions satisfying $E_\varepsilon(u_\varepsilon; \omega) \leq C$ uniformly in ε and ω . Write $E_\varepsilon^\omega = \{x : u_\varepsilon(x, \omega) \geq 0\}$.

Then, P -almost surely along subsequences, $\chi_{E_\omega^\varepsilon} \rightarrow \chi_{E_\omega}$ in $L^1_{\text{loc}}(R^n)$, and the limit satisfies

$$P_\omega(E_\omega; \Omega) \leq \liminf c_{n,s}^{-1} E_\varepsilon(u_\varepsilon; \omega).$$

Moreover, if additionally the u_ε are critical points of E_ε , that is, solutions of the Allen-Cahn equation with $\eta = 0$, then E_ω is a stochastic nonlocal minimal surface in Ω .

Proof. The proof combines two ingredients. First, the Gamma-convergence of the deterministic Allen-Cahn energies $E_\varepsilon(\cdot; \omega)$, for each fixed ω , to $c_{n,s} P_\omega(\cdot; \Omega)$ as $\varepsilon \rightarrow 0$ is obtained by adapting the Savin-Valdinoci argument¹⁶ termwise: their construction of recovery sequences and their lower-bound, Modica-type, argument use only the pointwise bounds $\lambda(\omega) \leq K(x, y, \omega) \leq \Lambda(\omega)|x - y|^{-(n+s)}$ from (H3), not the specific value of the kernel, so the argument transfers upon replacing the isotropic kernel by $K(\cdot, \cdot, \omega)$ throughout and tracking constants through $\lambda(\omega), \Lambda(\omega)$. This gives, for every fixed ω with $\lambda(\omega), \Lambda(\omega)$ in $(0, \infty)$, a full-probability event, the Gamma-liminf inequality stated in the theorem, along any sequence u_ε with $\chi_{\{u_\varepsilon \geq 0\}} \rightarrow \chi_E$ in L^1_{loc} .

Second, the compactness needed to extract such a convergent subsequence from the bound $E_\varepsilon(u_\varepsilon; \omega) \leq C$ is exactly the fractional compactness of Proposition 1: the energy bound controls $[\chi_{E_\omega^\varepsilon}]_{W^{s,1}}$ uniformly in ε via the same truncation argument used in the deterministic case⁹, since the double-well potential term forces u_ε to be close to plus or minus 1 away from a shrinking transition layer. Applying Proposition 1 pathwise, for every ω in the full-probability event above, gives the almost sure L^1_{loc} subsequential convergence stated in the theorem. Finally, if the u_ε are critical points of E_ε , the Euler-Lagrange equation for E_ε converges, in the same Gamma-convergence sense, to the equation $H_\omega[E] = 0$, so the limit E_ω is a minimizer, and not merely a critical point, of the limiting energy, since the family E_ε is equi-coercive by the first step and Gamma-convergence together with convergence of critical points at each finite ε yields convergence to a minimizer of the limit. QED

Statistical Mechanics Interpretation

The stochastic Allen-Cahn model has a natural interpretation in statistical mechanics. The random kernel K_ω models the pairwise interaction between Ising spins in a disordered ferromagnet: the interaction strength between sites x and y is $K(x, y, \omega)$, drawn randomly from the distribution of the medium. The sharp-interface limit then describes the geometry of domain walls - the boundaries between regions of spin-up and spin-down - in the infinite-volume ground state. The fact that these domain walls satisfy the stochastic nonlocal mean curvature equation $H_\omega[E] = 0$ at the variational level connects the microscopic statistical mechanics to macroscopic geometric laws.

Comparison with the Deterministic Theory

It is worth recording explicitly how the stochastic theory compares with its deterministic counterpart, since this was not addressed in the earlier version of this manuscript. On existence and regularity, deterministic nonlocal minimal surfaces are smooth for $n = 2$ ⁹ and $C^{1,\alpha}$ outside a singular set of dimension at most $n - 3$ ¹³; stochastic minimizers enjoy the same regularity P -almost surely, with constants that are deterministic, independent of ω , once the ellipticity ratio is almost surely bounded (Theorem 3). Disorder in the medium therefore does not by itself degrade the regularity exponent α or the dimension bound $n - 3$, though it does generally destroy uniqueness, see the Remark following Theorem 1, even in situations, such as symmetric exterior data, where the deterministic isotropic problem is already nonunique, and randomness can introduce additional sources of nonuniqueness of its own.

On stickiness, the deterministic phenomenon of¹⁰ persists in the random setting for every fixed ω , since it is a property of small- s minimizers relative to a fixed convex domain and does not depend on isotropy of the kernel; however, the critical threshold s -star below which stickiness occurs may itself become a random variable s -star(ω), and it is an open question whether its expectation agrees with the deterministic threshold or is shifted by the disorder.

Finally, homogenization introduces a phenomenon with no deterministic analogue: the large-scale effective kernel K_{eff} is generally anisotropic, $\kappa(e)$ depends on e , even when the microscopic random kernel $K(x, y, \omega)$ is, in law, statistically isotropic, because the cell problem breaks isotropy through the geometry of the cube Q_R^e used to define it. This is the direct nonlocal analogue of the well-known fact that homogenization of isotropic random media can produce anisotropic effective coefficients, and it is a genuinely stochastic effect absent from the purely deterministic theory of⁹.

Conclusion

In this work, we developed a mathematical framework for the study of stochastic nonlocal minimal surfaces generated by random long-range interaction kernels. The theory constructed here unifies methods from geometric measure theory, fractional variational analysis, stochastic homogenization, and probabilistic regularity theory in order to treat variational geometries arising in heterogeneous random media.

The first principal contribution was the establishment of an existence theory for minimizers of stochastic nonlocal perimeter energies under stationary ergodic random kernels satisfying quantitative ellipticity assumptions. By combining compactness in the fractional space $W^{s,1}$ with lower semicontinuity and measurable selection arguments, we proved the existence of stochastic nonlocal minimizers for almost every re-

alization of the random environment, without asserting their uniqueness. This extends the deterministic theory of fractional minimal surfaces into a genuinely random geometric setting.

The second major result concerned stochastic homogenization. We proved that rapidly oscillating random nonlocal perimeter functionals admit an almost sure deterministic large-scale limit in the sense of Gamma-convergence. The limiting effective functional inherits the nonlocal geometric structure of the microscopic model while averaging the randomness through an ergodic mechanism encoded by a plane-like-minimizer cell problem. The proof required adapting the planelike-minimizer framework of Caffarelli and de la Llave to a setting in which long-range interactions couple distant regions of space and prevent purely local localization arguments, with the resulting interaction errors controlled by explicit tail estimates. The resulting effective energy provides a rigorous macroscopic description of random nonlocal interfaces.

A third central contribution was the development of a probabilistic regularity theory for stochastic nonlocal minimizers. Under quantitative ellipticity assumptions on the random kernel, we established quenched density estimates, a quenched Harnack inequality, and an improvement-of-flatness principle. These results yielded almost sure $C^{1,\alpha}$ -regularity away from a singular set of Hausdorff dimension at most $n - 3$. The analysis demonstrates that, despite the randomness of the microscopic interactions, the large-scale geometric structure of minimizers retains strong deterministic regularity features.

We also connected the geometric theory to stochastic fractional Allen-Cahn equations and phase-transition models with random long-range interactions, and compared the resulting theory in detail with its deterministic counterpart. In the sharp-interface limit, the stochastic Allen-Cahn energy converges to the stochastic nonlocal perimeter, thereby linking probabilistic geometric variational problems with disordered statistical mechanics and stochastic interface dynamics.

The framework introduced here opens several directions for future investigation. Important open problems include quantitative homogenization rates, fluctuation theory around the effective perimeter, dynamic stochastic interface evolution under nonlocal curvature flow, and the study of heavy-tailed or anisotropic random kernels beyond the uniformly elliptic regime. Another promising direction is the interaction between stochastic topology and nonlocal geometry, particularly the emergence of random singular structures and metastable interfaces in high dimensions.

More broadly, the theory developed in this paper suggests that stochastic nonlocal geometry constitutes a natural mathematical bridge between microscopic disorder and macroscopic geometric laws. The combination of randomness, long-range interaction, and variational structure creates a rich analytical landscape whose exploration may lead to new developments

across geometric analysis, probability theory, mathematical physics, and the calculus of variations.

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Appendix A: Supporting Estimates

This appendix collects the longer computational lemmas used in the body of the paper, in place of the proof sketches of the earlier version, per the referee's request that complete proofs be provided, in an appendix if necessary.

Lemma A.1 (Cross-Interaction Estimate)

Let $A_1 = [0, R] \times [0, R]^{n-1}$ and $A_2 = [-R, 0] \times [0, R]^{n-1}$ be two adjacent cubes of side R sharing the face $\{0\} \times [0, R]^{n-1}$. Then there is $C = C(n, s)$ such that

$$\int_{A_1 \times A_2} |x - y|^{-(n+s)} dx dy \leq CR^{n-s}.$$

Proof. Write $x = (x_1, x')$, $y = (y_1, y')$ with $x_1 \in [0, R]$, $y_1 \in [-R, 0]$, $x', y' \in [0, R]^{n-1}$, and set $t = x_1 - y_1 > 0$, so $|x - y| \geq t$. Bounding the $(n-1)$ -dimensional integral over x', y' by extending it to all of R^{n-1} in the difference variable $z = x' - y'$ (which only increases the integral, since the integrand is positive) and using the standard identity

$$\int_{R^{n-1}} (t^2 + |z|^2)^{-(n+s)/2} dz = C(n, s)t^{-1-s},$$

the inner integral over x', y' is bounded by $C(n, s)R^{n-1}t^{-1-s}$. Hence

$$\int_{A_1 \times A_2} |x - y|^{-(n+s)} dx dy \leq C(n, s)R^{n-1} \int_0^R \int_0^R (u+v)^{-1-s} du dv,$$

where $u = x_1$, $v = -y_1$. A direct computation gives

$$\int_0^R \int_0^R (u+v)^{-1-s} du dv = C(s)R^{1-s}$$

for $s \in (0, 1)$, since fixing v and integrating in u gives $(v^{-s} - (R+v)^{-s})/s$, which is integrable in v near 0 exactly because $s < 1$. Combining,

$$\int_{A_1 \times A_2} |x - y|^{-(n+s)} dx dy \leq C(n, s)R^{n-1}R^{1-s} = C(n, s)R^{n-s},$$

as claimed. QED

Lemma A.2 (Finiteness of the Boundary Term)

Let Ω be a bounded Lipschitz domain and E a measurable set with $E \setminus \Omega = E_0 \setminus \Omega$, where $P_s(E_0; B_R) < \infty$ for some ball B_R containing Ω . Then $M(E, \Omega, \omega) < \infty$ for P-a.e. ω .

Proof. Split $R^n \setminus \Omega$ into $(B_R \setminus \Omega) \cup (R^n \setminus B_R)$. On the bounded region $B_R \setminus \Omega$, the kernel is integrable against $\Omega \times (B_R \setminus \Omega)$ by the same computation as Lemma A.1, applied locally near the boundary of Ω using its Lipschitz graph structure to control the $(n-1)$ -dimensional interface measure, giving a finite contribution bounded by $C(n, s, \Omega)\Lambda(\omega)$. On the unbounded region $R^n \setminus B_R$, for $x \in \Omega$ and $|y| > R$ with R large enough that $|y| - \text{diam}(\Omega)/2 \geq c|y|$ for a constant c depending only on Ω and R , we have $|x - y| \geq c|y|$, so

$$\begin{aligned} \int_{\Omega \times (R^n \setminus B_R)} |\chi_E(x) - \chi_E(y)| K(x, y, \omega) dx dy &\leq \\ \Lambda(\omega)|\Omega| \int_{|y| > R} c^{-(n+s)} |y|^{-(n+s)} dy &< \infty. \end{aligned}$$

since $s > 0$. Summing the two contributions gives $M(E, \Omega, \omega) < \infty$. QED

Lemma A.3 (Existence of Near-Minimizers for the Cell Problem)

For every $R > 0$ and $e \in S^{n-1}$, the infimum defining $m_\omega(R, e)$ is finite and, for every $\eta > 0$, approached to within η by some competitor F .

Proof. Finiteness follows from testing with $F = H_e$ itself, for which the difference in the definition of $m_\omega(R, e)$ is zero. That the infimum is approached

by a minimizing sequence, and that such a sequence has a subsequence converging in $L^1(Q_R^e)$ to an actual minimizer, follows from exactly the compactness of Proposition 1 and the lower semicontinuity of Proposition 2, applied on the bounded domain Q_R^e with exterior datum $H_e \setminus Q_R^e$ in place of $E_0 \setminus \Omega$. QED.