Polygons with a Fixed Perimeter and the Maximization of Area

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The correlation between shapes, perimeter, and area is crucial in the study of shapes, polygons, and optimization. Historically, its importance can be observed in the classical Isoperimetric Problem, which inquired the maximum area of a shape with a fixed perimeter and was later proven and demonstrated using the isoperimetric inequality, and the tale of Queen Dido, which involves her clever method of maximizing the amount of land with a minimal boundary. In addition, it can provide a deeper conceptual understanding and mathematical knowledge. Although the isoperimetric inequality provides a deep conceptual understanding of shapes and optimization, to further the enrichment of mathematical knowledge and conceptual understanding, this study focuses on the optimization of polygons. This study aims to determine whether there can or cannot exist a polygon, amongst every polygon with the same perimeter, which has the largest area. Through this, it was concluded that there cannot exist such a polygon with a fixed perimeter that had the largest possible area. Instead, it was also concluded that as the area of polygons with the same perimeter approached its upper bound, it approached the shape of a circle with the same perimeter.

Keywords: Convex, Polygon, Convex Hull, Fixed Perimeter, Scaling, Equilateral, Equiangular, Largest Area, Polygons with a Set Perimeter, Base Polygons, *n*-sided, Derivative

Introduction

The relationship between the area and the perimeter of a polygon is one of the most important concepts in the study of polygons. Its analysis is not only important in mathematics, but it is also historically important, as observed in the classical Isoperimetric Problem[1], which considers how to maximize the area of a shape with a fixed perimeter. The tale of Queen Dido is an early example of this applied practically, as she found a clever way to maximize the amount of land with a minimal boundary[2]. Mathematical knowledge is also of great importance in this topic as students can learn valuable insights. For instance, Heron's formula, the formula for the area of triangles based on the length of its sides, is an alternative of the formula which utilizes the altitude and the length of its base to find the area, offering the students a deeper, conceptual understanding of mathematics[3]. By asking students to research the effect of perimeter manipulation (or diameter manipulation) on the properties of different polygons (i.e., triangles, quadrilaterals, and pentagons), one not only adds richness to mathematical reasoning, but also will help students think more about problem-solving and build a stronger base in geometry[3]. The isoperimetric inequality states that $4\pi A \leq P^2$, where A and P denote the area and perimeter of a shape, respectively, in which equality holds if and only if the shape is a circle[4]. This implies that, of all the shapes with the same perimeter, the one which has the shape of a circle has the largest area. Despite the isoperimetric inequality truly providing a great understanding of shapes and optimization, to aid in the enrichment of mathematical knowledge, the aim of this study is to focus on polygons and to determine whether or not there existed the polygon with a fixed perimeter that has the maximum area.

The Conjecture

This study aims to prove whether or not there exists a polygon with the largest area amongst every polygon which has the same perimeter. In order for this to be done, the conjecture below must be either proven true or false:

Conjecture 0.0 Suppose there exists a set of every polygon with the same perimeter. Within this set, there exists a polygon with the largest area.

The remainder of this study demonstrates that Conjecture 0.0 is incorrect by first assuming the existence of a polygon which has the largest area within this set and then proving that it cannot actually exist.

The Definitions

Throughout this study, there are many terms that must be defined.

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n-sided

This is defined as having n sides, where n represents a positive integer.

Equivalence Classes under Similarity

The set of all shapes which have the same shape, regardless of their size, position, or orientation. Hence, the set of shapes that are all similar to one another[5].

Convex Polygon

This is defined as a polygon whose interior angles are all smaller than 180[6].

Convex Hull

The smallest convex set that entirely contains a given set of points or a shape[7].

Polygons with a Set Perimeter

This is defined as a set of all polygons with the same perimeter.

The Largest Polygon with a Set Perimeter

This is defined as the polygon with a set perimeter that has the largest area.

Base Polygon

This is defined as a polygon with a perimeter of 1.

The Largest Base Polygon

This is defined as the base polygon with the largest area.

The Proof

Scaling Polygons and Base Polygons

When finding the largest polygon with a set perimeter, the perimeter is a major factor that must be considered. However, it is possible to find the largest polygon with a set perimeter through the scaling of base polygons. Every base polygon can be scaled to match every corresponding similar polygon with a set perimeter.

Suppose there exists a n-sided convex polygon, denoted polygon $A_1A_2\ldots A_n$, and an m-sided convex polygon, denoted polygon $B_1B_2\ldots B_m$, where $2 < n$ and $2 < m$.

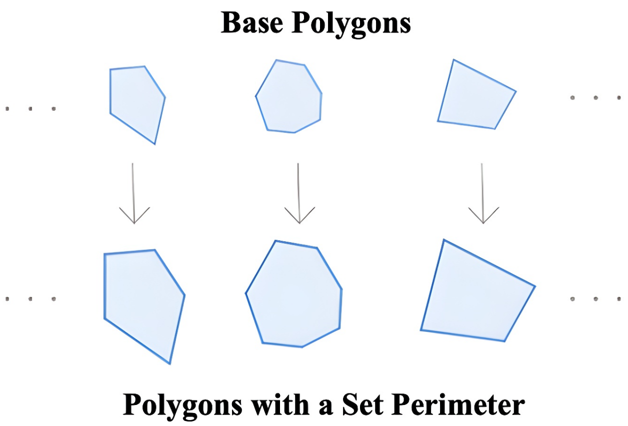

Fig. 1 A schematic illustrating that every base polygon can be scaled to match every similar polygon with a set perimeter

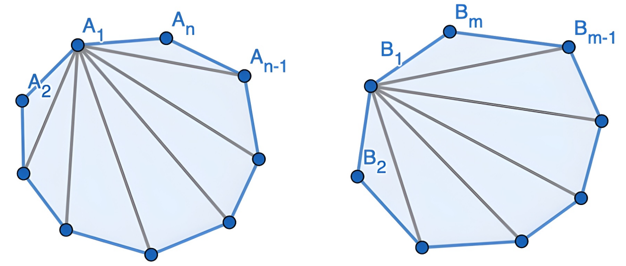

Fig. 2 A visual representation of polygon $A_1A_2\ldots A_n$ and polygon $B_1B_2\ldots B_m$, where points $A_1, A_2, A_{n-1}, A_n, B_1, B_2, B_{m-1}$, and B_m are labeled and the diagonals from points A_1 and B_1 to other points within polygon $A_1A_2\ldots A_n$ and polygon $B_1B_2\ldots B_m$ are illustrated, respectively

Let A and B denote the area of polygon $A_1A_2\ldots A_n$ and polygon $B_1B_2\ldots B_m$, respectively. Additionally, let the perimeter of both polygons be 1. In addition, suppose A is larger than B. Using the sine rule for area[8], the equations of A and B are shown below:

$$A = \sum_{k=2}^{n-1} \frac{1}{2}(A_1A_k)(A_1A_{k+1})\sin\angle A_kA_1A_{k+1}$$

$$B = \sum_{k=2}^{m-1} \frac{1}{2}(B_1B_k)(B_1B_{k+1})\sin\angle B_kB_1B_{k+1}$$

Suppose there exists an n-sided convex polygon, denoted polygon $C_1C_2\ldots C_n$, which is an element of the equivalence class under similarity which includes polygon $A_1A_2\ldots A_n$ and has a perimeter of λ, where $0 < \lambda$. In addition, let the length of C_iC_j be λ times the length of A_iA_j, for unequal arbitrary integers i and j which are greater than 0 and less than $n+1$. As polygon $C_1C_2\ldots C_n$ is similar to polygon $A_1A_2\ldots A_n$, $\angle C_iC_jC_k$ must be equal to $\angle A_iA_jA_k$, for unequal arbitrary integers i, j, and k which are greater than 0 and less than $n+1$.

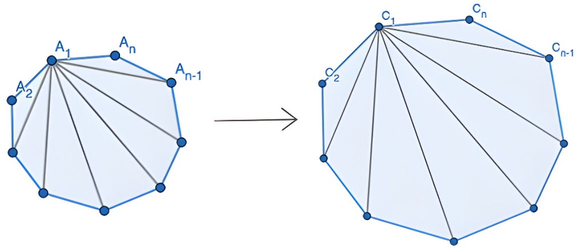

Fig. 3 A visual representation of polygon $A_1A_2\ldots A_n$ scaling to match polygon $C_1C_2\ldots C_n$, where points $A_1, A_2, A_{n-1}, A_n, C_1, C_2, C_{n-1}$, and C_n are labeled and the diagonals from points A_1 and C_1 to other points within polygon $A_1A_2\ldots A_n$ and polygon $C_1C_2\ldots C_n$ are illustrated, respectively

Suppose there also exists an m-sided convex polygon, denoted polygon $D_1D_2\ldots D_m$, which is an element of the equivalence class under similarity which includes polygon $B_1B_2\ldots B_m$ and also has a perimeter of λ. In addition, let the length of D_iD_j be λ times the length of B_iB_j, for unequal arbitrary integers i and j which are greater than 0 and less than $m+1$. Similarly, as polygon $D_1D_2\ldots D_m$ is similar to polygon $B_1B_2\ldots B_m$, $\angle D_iD_jD_k$ must be equal to $\angle B_iB_jB_k$, for unequal arbitrary integers i, j, and k which are greater than 0 and less than $m+1$.

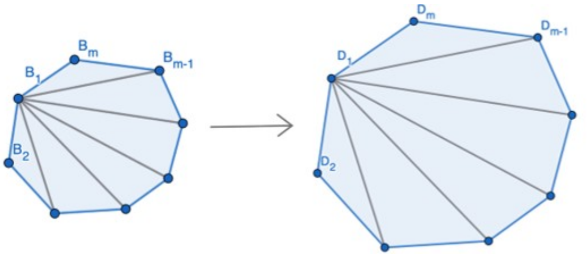

Fig. 4 A visual representation of polygon $B_1B_2\ldots B_m$ scaling to match polygon $D_1D_2\ldots D_m$, where points $B_1, B_2, B_{m-1}, B_m, D_1, D_2, D_{m-1}$, and D_m are labeled and the diagonals from points B_1 and D_1 to other points within polygon $B_1B_2\ldots B_m$ and polygon $D_1D_2\ldots D_m$ are illustrated, respectively

Let C and D denote the area of polygon $C_1C_2\ldots C_n$ and polygon $D_1D_2\ldots D_m$, respectively. Using the sine rule for area[8], the equations of C and D are shown below:

$$C = \sum_{k=2}^{n-1} \frac{1}{2}(C_1C_k)(C_1C_{k+1}) \sin\angle C_kC_1C_{k+1}$$

$$D = \sum_{k=2}^{m-1} \frac{1}{2}(D_1D_k)(D_1D_{k+1}) \sin\angle D_kD_1D_{k+1}$$

These equations can be simplified as shown below:

$$\begin{aligned} C &= \sum_{k=2}^{n-1} \frac{1}{2}(C_1C_k)(C_1C_{k+1}) \sin\angle C_kC_1C_{k+1} \\ &= \sum_{k=2}^{n-1} \frac{1}{2}(\lambda A_1A_k)(\lambda A_1A_{k+1}) \sin\angle A_kA_1A_{k+1} \\ &= \lambda^2 \sum_{k=2}^{n-1} \frac{1}{2}(A_1A_k)(A_1A_{k+1}) \sin\angle A_kA_1A_{k+1} \\ &= \lambda^2 A \end{aligned}$$

$$\begin{aligned} D &= \sum_{k=2}^{m-1} \frac{1}{2}(D_1D_k)(D_1D_{k+1}) \sin\angle D_kD_1D_{k+1} \\ &= \sum_{k=2}^{m-1} \frac{1}{2}(\lambda B_1B_k)(\lambda B_1B_{k+1}) \sin\angle B_kB_1B_{k+1} \\ &= \lambda^2 \sum_{k=2}^{m-1} \frac{1}{2}(B_1B_k)(B_1B_{k+1}) \sin\angle B_kB_1B_{k+1} \\ &= \lambda^2 B \end{aligned}$$

As A is larger than B and λ is greater than 0, $\lambda^2 A$ is greater than $\lambda^2 B$. Hence, C is greater than D.

For polygons that are not convex, a similar method can be used. Suppose E is a base polygon which is not convex. E can be divided into p amount of triangles, denoted $E_1, \ldots, E_p$[9]. Therefore, the area of E is the sum of the areas of $E_1, \ldots, E_p$.

Suppose F is a polygon with a perimeter of λ and is an element of the equivalence class under similarity which includes E. In addition, let $F_1, \ldots, F_p$ be triangles which are an element of the equivalence class under similarity containing $E_1, \ldots, E_p$, respectively. In addition, for every integer value of i between 1 and p, let the perimeter of F_i be λ times the perimeter of E_i. Similarly to how E was divided into $E_1, \ldots, E_p$, F can also be divided into $F_1, \ldots, F_p$. Therefore, the area of F is the sum of the areas of $F_1, \ldots, F_p$.

Similarly to as shown above, for every integer value of i between 1 and p, the area of F_i is λ^2 times the area of E_i. Therefore, the area of F is λ^2 times the area of E. Suppose G is a base polygon which has a smaller area than E and H is a polygon with a perimeter of λ and is an element of the equivalence class under similarity which includes G. The area of H would be λ^2 times the area of G similarly to how the area of F is λ^2 times the area of E. As the area of E is greater than the area of G, the area of F must be greater than the area of H.

Therefore, if a base polygon has a greater area than another base polygon, a polygon that is similar to the former base polygon must have a greater area than a polygon with the same perimeter that is similar to the latter base polygon.

Furthermore, it can be proven that the largest polygon with a set perimeter is similar to the largest base polygon. The

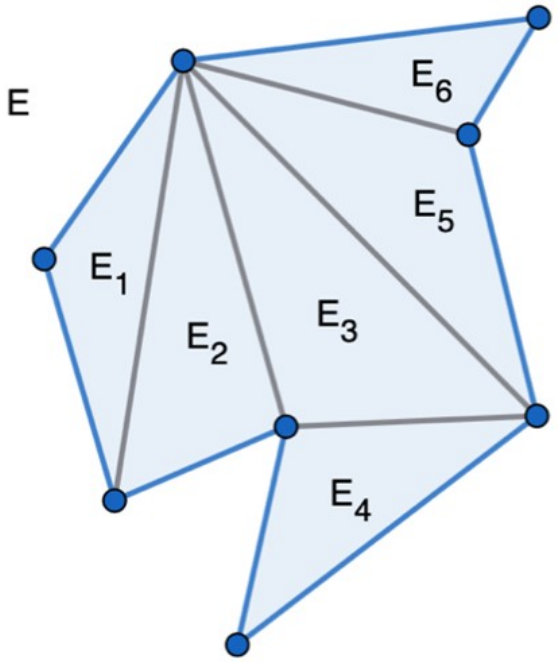

Fig. 5 A visual representation of the polygon E and an illustration, where p is equal to 6, of the division of E into 6 triangles, labeled E_1, E_2, E_3, E_4, E_5, and E_6

largest base polygon has a greater area than every other base polygon. Let E_0 denote the equivalence class under similarity which contains the largest base polygon. As every base polygon can be scaled to match every polygon with a set perimeter that is an element of the same equivalence class under similarity, the polygon with a set perimeter that is an element of E_0 must have a larger area compared to the polygons with the same perimeter which are not elements of E_0. In short, the largest polygon with a set perimeter must be an element of the equivalence class under similarity which contains the largest base polygon.

Thus, if it can be proven that the largest base polygon does not exist, there cannot exist an equivalence class under similarity which contains the largest base polygon. Furthermore, this would also prove that the largest polygon with a set perimeter cannot exist as well, disproving Conjecture 0.0. Hence, the remainder of this paper intends to demonstrate that the largest base polygon cannot exist in order to disprove Conjecture 0.0.

The Method of Elimination

There are an infinite number of base polygons that can have different side lengths, interior angles, and number of sides.

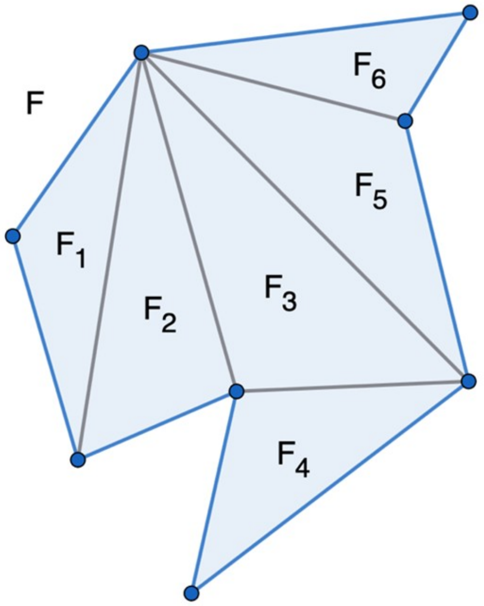

Fig. 6 A visual representation of the polygon F and an illustration, where p is equal to 6, of the division of F into 6 triangles, labeled F_1, F_2, F_3, F_4, F_5, and F_6

Comparing the area of every base polygon to one another to determine the largest base polygon is not a feasible method because of this. For this reason, there must be a more efficient method to distinguish the polygons that cannot be the largest base polygon.

Theorem 0.1 Suppose there is a set, denoted as X, which consists of all the n-sided base polygons that satisfy a certain property. If it is possible to prove the existence of a n-sided base polygon with a larger area for each element of X using this property, it can be proven that the largest n-sided base polygon does not have this property.

Suppose X is the set of all the n-sided base polygons that satisfy a certain property. Additionally, for every element of X, suppose this property can be utilized to prove the existence of a n-sided base polygon that has a larger area. Consequently, every element of X cannot be the largest n-sided base polygon because there exists a n-sided base polygon with a larger area. Correspondingly, the largest base polygon cannot be an element of X. As the largest base polygon is not an element of X, it must not satisfy this certain property, proving Theorem 0.1.

In the remainder of this paper, it is proven that the largest base polygon cannot possess certain properties by proving that, for each base polygon with those properties, there exists a base polygon with a larger area. Hence, Theorem 0.1 can be used to efficiently narrow down the possibilities through the elimination of every base polygon that satisfies certain properties that prevents them from being the largest base polygon.

Non-Convex Polygons

Theorem 1.0 The largest n-sided base polygon is convex.

Let polygon $Q_1Q_2\dots Q_n$ be a n-sided base polygon which is not convex. n must be more than 3 because all triangles are convex as the three interior angles of a triangle are all less than 180[10]. $(3 < n)$

In addition, suppose the convex hull of polygon $Q_1Q_2\dots Q_n$ is denoted as Q_0. Let Q_i and Q_j be two different vertices of Q_0 which are connected to each other by an edge of Q_0. However, suppose Q_i and Q_j are not connected by an edge within polygon $Q_1Q_2\dots Q_n$. Hence, j must be greater than $i+1$. $(0 < i < i+1 < j < n+1)$

As Q_i and Q_j are vertices of Q_0 and Q_i and Q_j are connected by an edge of Q_0, points $Q_{i+1},\dots,Q_{j-1}$ must be within the interior of Q_0.

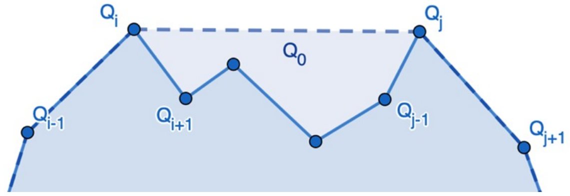

Fig. 7 A visual representation of polygon $Q_1Q_2\dots Q_n$ that focuses on vertices $Q_{i-1}, Q_i, Q_{i+1}, Q_{j-1}, Q_j$, and Q_{j+1} and includes the convex hull of polygon $Q_1Q_2\dots Q_n$, which is denoted as Q_0

Let points $R_{i+1},\dots,R_{j-1}$ be the reflection of points $Q_{i+1},\dots,Q_{j-1}$ across line Q_iQ_j, respectively. In addition, let R_1 denote polygon $Q_iR_{i+1}\dots R_{j-1}Q_j$.

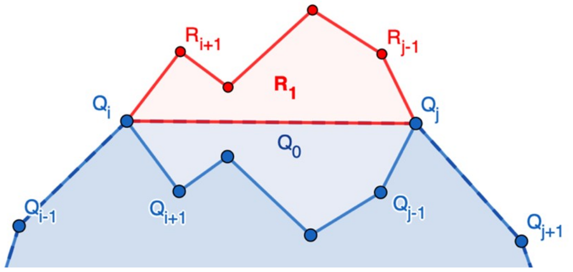

Fig. 8 A visual representation of polygon $Q_1Q_2\dots Q_n$ that focuses on vertices $Q_{i-1}, Q_i, Q_{i+1}, Q_{j-1}, Q_j$, and Q_{j+1}, includes the convex hull of polygon $Q_1Q_2\dots Q_n$, which is denoted as Q_0, and illustrates polygon R_1, where points R_{i+1} and R_{j-1} are labeled

Points $Q_1,\dots,Q_n$ excluding points Q_i and Q_j are on the opposite side of points $R_{i+1},\dots,R_{j-1}$ across line Q_iQ_j as Q_0, which contains points $Q_1,\dots,Q_n$, are all on the same side of line Q_iQ_j, excluding points Q_i and Q_j. Therefore, polygon $Q_iR_{i+1}\dots R_{j-1}Q_j$ and polygon $Q_1Q_2\dots Q_n$ meet at common vertices Q_i and Q_j.

Let P_q, A_q, P_r, and A_r denote the perimeter and area of polygon $Q_1\dots Q_iQ_j\dots Q_n$ and the perimeter and area of R_1, respectively. As points $R_{i+1},\dots,R_{j-1}$ are the reflection of points $Q_{i+1},\dots,Q_{j-1}$ across line Q_iQ_j, respectively, the perimeter and area of polygon $Q_iQ_{i+1}\dots Q_{j-1}Q_j$ must be the same as the perimeter and area of R_1, which is P_r and A_r, respectively. Points $Q_{i+1},\dots,Q_{j-1}$ are within the interior of Q_0. Therefore, the area of polygon $Q_1Q_2\dots Q_n$ is the area of polygon $Q_1\dots Q_iQ_j\dots Q_n$ subtracted by the area of $Q_iQ_{i+1}\dots Q_{j-1}Q_j$, which is $A_q - A_r$. Let R_0 denote polygon $Q_1\dots Q_iR_{i+1}\dots R_{j-1}Q_j\dots Q_n$. R_0 and polygon $Q_1\dots Q_iQ_j\dots Q_n$ share the same edge, which is Q_iQ_j. Therefore, the area of R_0 is the area of polygon $Q_1\dots Q_iQ_j\dots Q_n$ plus the area of R_1, which is $A_q + A_r$. Thus, R_0 has a larger area than polygon $Q_1Q_2\dots Q_n$. In addition, the perimeter of polygon $Q_1Q_2\dots Q_n$ can be represented as $P_q - Q_iQ_j + P_r - Q_iQ_j$. Similarly, the perimeter of R_0 can be represented as $P_q - Q_iQ_j + P_r - Q_iQ_j$. As polygon $Q_1Q_2\dots Q_n$ is a base polygon and has the same perimeter as R_0, R_0 must have a perimeter of 1.

In conclusion, for every n-sided base polygon which is not convex, there exists a n-sided base polygon which has a larger area. Utilizing Theorem 0.1, it can be proven that the largest n-sided base polygon is convex, proving Theorem 1.0.

Triangles and Equilateral Polygons

Theorem 1.1 Suppose there is a non-equilateral triangle ABC in which AB and AC are unequal. In addition to this triangle, there exists an isosceles triangle in which the base of the isosceles triangle has the length of BC and the two equal sides have the length of the mean of the lengths of AB and AC. It can be proven that this isosceles triangle has a larger area than $\triangle ABC$.

Suppose there is a triangle ABC with a perimeter of p. Within $\triangle ABC$, BC has a length of a, the length of AB is denoted as x, and $\angle ABC$ has an angle of θ radians. In addition, a and p are constants $(0 < p,\ 0 < a,\ 0 < x,\ 0 < \theta)$.

The triangle inequality states that the sum of the length of any two arbitrary sides of a triangle must be greater than the length of the remaining side[11]. Because of this, the following inequalities can be formed as shown below:

$$AB < BC + AC$$
$$AC < AB + BC$$
$$BC < AB + AC$$

As the perimeter of $\triangle ABC$, the length of AB, and the length of BC are p, x, and a, respectively, the length of AC must be

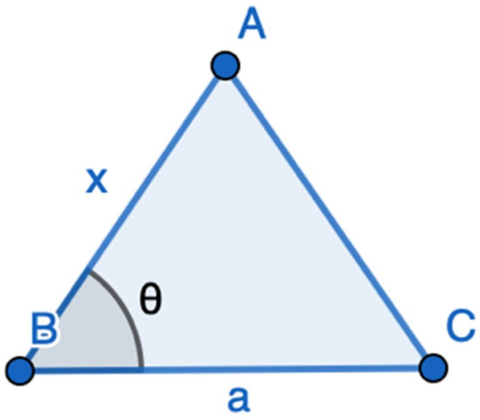

Fig. 9 A visual representation of $\triangle ABC$ that includes $\angle ABC$, which has an angle of θ radians, the length of BC, which is a, and the length of AB, which is x

$p-a-x$, which must also be positive. Using this, the three inequalities can be altered and simplified as shown below:

$$\begin{aligned}
&AB < BC + AC &&\Rightarrow\; x < a + p - a - x &&\Rightarrow\; 2x < p &&\Rightarrow\; x < \tfrac{1}{2}p\\
&AC < AB + BC &&\Rightarrow\; p - a - x < x + a &&\Rightarrow\; p - 2a < 2x &&\Rightarrow\; \tfrac{p-2a}{2} < x\\
&BC < AB + AC &&\Rightarrow\; a < x + p - a - x &&\Rightarrow\; 2a < p &&\Rightarrow\; a < \tfrac{1}{2}p
\end{aligned}$$

Therefore, a is greater than 0 and less than $\frac{1}{2}p$ and x is greater than $\frac{p-2a}{2}$ and less than $\frac{1}{2}p$. In addition, as $\triangle ABC$ is a triangle, $\angle ABC$ must be less than 180. Thus, θ must be less than π.

Let $S(x)$ be the function for the area of $\triangle ABC$. $S(x)$ can be written as shown below:

$$S(x) = \frac{1}{2}ax\sin\theta$$

The absolute value of x, $|x|$, is defined as follows[12]:

$$|x| = \begin{cases} x & \text{if } x \geq 0\\ -x & \text{if } x < 0\end{cases}$$

As the square root function always gives non-negative values for real numbers, the value of $\sqrt{x^2}$ is x when x is more than or equal to 0 and $-x$ when x is less than 0. Therefore, $\sqrt{x^2}$ is equal to $|x|$. As $\sqrt{1-\cos^2\theta}$ can be simplified to $\sqrt{\sin^2\theta}$, it is equal to $|\sin\theta|$. As the sine of θ is positive because θ is greater than 0 and less than π, $|\sin\theta|$ is equal to the sine of θ. Therefore, $\sqrt{1-\cos^2\theta}$ is equal to the sine of θ. Thus, $S(x)$ can be altered as shown below:

$$S(x) = \frac{1}{2}ax\sin\theta = \frac{1}{2}ax\sqrt{1-\cos^2\theta}$$

To represent the area of $\triangle ABC$ as a function of x, the cosine of θ must also be represented as a function of x. Using the law of cosines[8], the cosine of θ can be written as shown below:

$$\cos\theta = \frac{a^2 + x^2 - (p-a-x)^2}{2ax}$$

As the cosine of θ can be represented as a function of x, the formula for the area of $\triangle ABC$ can also be written as a function of x as shown below:

$$\begin{aligned}
S(x) = \frac{1}{2}ax\sqrt{1-\cos^2\theta} &= \frac{1}{2}ax\sqrt{1-\left(\frac{a^2+x^2-(p-a-x)^2}{2ax}\right)^2}\\
&= \frac{1}{4}\sqrt{(2ax)^2 - \left(a^2+x^2-(p-a-x)^2\right)^2}\\
&= \frac{1}{4}\sqrt{(2ax)^2 - \left(a^2+x^2-(p^2+a^2+x^2+2ax-2pa-2px)\right)^2}\\
&= \frac{1}{4}\sqrt{(2ax)^2 - (-p^2-2ax+2pa+2px)^2}\\
&= \frac{1}{4}\sqrt{(-p^2+2pa+2px)(p^2+4ax-2pa-2px)}\\
&= \frac{1}{4}\sqrt{p(-p+2a+2x)(p-2a)(p-2x)}\\
&= \frac{1}{4}\sqrt{p(p-2a)}\sqrt{(-p+2a+2x)(p-2x)}\\
&= \frac{1}{4}\sqrt{p(p-2a)}\sqrt{-4x^2+4(p-a)x-p(p-2a)}\\
&= \frac{1}{4}\sqrt{p(p-2a)}\sqrt{-4\left(x^2-(p-a)x+\left(\tfrac{p-a}{2}\right)^2\right)+a^2}\\
&= \frac{1}{4}\sqrt{p(p-2a)}\sqrt{-4\left(x-\tfrac{p-a}{2}\right)^2+a^2}
\end{aligned}$$

For the value of x in which $S(x)$ has the maximum value, $\sqrt{-4\left(x-\frac{p-a}{2}\right)^2+a^2}$ must also have its maximum value because $\frac{1}{4}\sqrt{p(p-2a)}$ is constant. Using the bounds of x, the following inequalities can be proven:

$$\begin{gathered}
\frac{p-2a}{2} < x < \frac{1}{2}p\\
-\frac{1}{2}a < x - \frac{p-a}{2} < \frac{1}{2}a\\
0 \leq \left(x-\frac{p-a}{2}\right)^2 < \frac{1}{4}a^2\\
0 < -4\left(x-\frac{p-a}{2}\right)^2 + a^2 \leq a^2\\
0 < \sqrt{-4\left(x-\frac{p-a}{2}\right)^2 + a^2} \leq a
\end{gathered}$$

Thus, $\sqrt{-4\left(x-\frac{p-a}{2}\right)^2+a^2}$ has a maximum of a, which is when x is equal to $\frac{p-a}{2}$ because $-4\left(x-\frac{p-a}{2}\right)^2$ would be equal to 0, allowing $\sqrt{-4\left(x-\frac{p-a}{2}\right)^2+a^2}$ to equal $\sqrt{a^2}$ which is equal to a. Therefore, the following inequality in which $S(x)$ has the maximum value when x is equal to $\frac{p-a}{2}$ can be shown below:

$$S(x)=\frac{1}{4}\sqrt{p(p-2a)}\sqrt{-4\left(x-\frac{p-a}{2}\right)^2+a^2}\leq\frac{1}{4}a\sqrt{p(p-2a)}$$

$$\left(\text{Equal when } x=\frac{p-a}{2}\right)$$

Hence, when x is $\frac{p-a}{2}$, $\triangle ABC$ has a maximum area of $\frac{1}{4}a\sqrt{p(p-2a)}$. Furthermore, when x is $\frac{p-a}{2}$, AB and AC both have equal lengths of $\frac{p-a}{2}$, causing it to be an isosceles triangle. Thus, if there is a non-equilateral triangle ABC in which AB and AC are not equal, there exists an isosceles triangle with the same perimeter as $\triangle ABC$ and a base with the length of BC which has a larger area than $\triangle ABC$, proving Theorem 1.1.

Theorem 1.2 The largest n-sided base polygon is a convex equilateral n-sided base polygon.

Theorem 1.1 demonstrates that there exists a 3-sided base polygon with a larger area for every non-equilateral 3-sided base polygon. To prove the existence of a n-sided base polygon with a larger area for every non-equilateral n-sided base polygon with at least 4 sides, let's suppose that there is a convex non-equilateral n-sided base polygon, referred to as polygon $A_1A_2\ldots A_n$, in which A_iA_{i+1} and $A_{i+1}A_{i+2}$ are two arbitrary adjacent sides that do not have the same length ($4\leq n$, $1\leq i\leq n-2$).

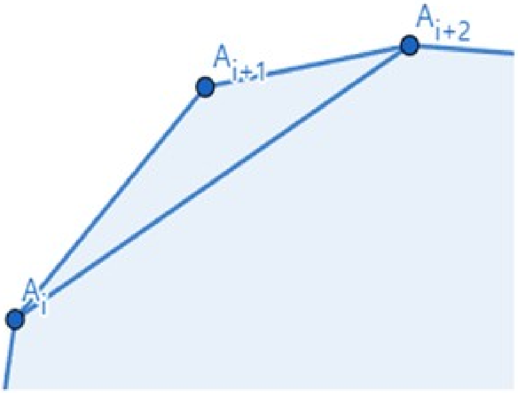

Fig. 10 A visual representation of polygon $A_1A_2\ldots A_n$ that focuses on vertices A_i, A_{i+1}, and A_{i+2} and includes A_iA_{i+2}

To prove that there exists a n-sided base polygon that has a larger area compared to polygon $A_1A_2\ldots A_n$, polygon $A_1A_2\ldots A_n$ must first be divided into two separate polygons, triangle $A_iA_{i+1}A_{i+2}$ and polygon $A_1A_2\ldots A_iA_{i+2}\ldots A_n$. As A_iA_{i+1} and $A_{i+1}A_{i+2}$ do not have the same lengths, using Theorem 1.1, it can be proven that there exists an isosceles triangle with a larger area which has a base with the length of A_iA_{i+2} and the same perimeter as $\triangle A_iA_{i+1}A_{i+2}$. Let's denote this isosceles triangle as triangle ABC in which AC is the base of the isosceles triangle. A new n-sided polygon can be composed by shifting the position and rotating $\triangle ABC$ so that AC coincides with A_iA_{i+2} and vertex B is not on the interior or the boundary of polygon $A_1A_2\ldots A_iA_{i+2}\ldots A_n$. The new polygon can be referred to as polygon $A_1A_2\ldots A_iBA_{i+2}\ldots A_n$. As $\triangle ABC$ has a larger area compared to $\triangle A_iA_{i+1}A_{i+2}$, polygon $A_1A_2\ldots A_iBA_{i+2}\ldots A_n$ has a larger area compared to polygon $A_1A_2\ldots A_n$. In addition, the sum of the lengths of A_iA_{i+1} and $A_{i+1}A_{i+2}$ is equal to the sum of the lengths of AB and BC because the perimeter of both polygons is equal and A_iA_{i+2} and AC both have the same lengths. This reveals that polygon $A_1A_2\ldots A_iBA_{i+2}\ldots A_n$ is also a base polygon because it has the same perimeter as polygon $A_1A_2\ldots A_n$.

In short, for every convex non-equilateral n-sided base polygon, there exists a n-sided base polygon with a larger area. Theorem 1.0 states that the largest n-sided base polygon must be convex. Thus, using Theorem 0.1, it can be proven that the largest n-sided base polygon is equilateral. Hence, the largest n-sided base polygon is a convex equilateral n-sided base polygon, proving Theorem 1.2.

Quadrilaterals and Equiangular Polygons

Theorem 2.1 Suppose there is a convex quadrilateral $ABCD$ in which AB, CD, and AD all have the same length and $\angle BAD$ and $\angle CDA$ do not have the same angles. In addition to $ABCD$, there exists a convex quadrilateral, denoted $WXYZ$, in which the lengths of WX, YZ, and WZ all equal the length of AB, the length of XY equals the length of BC, and $\angle XWZ$ has the same angle as $\angle YZW$. It can be proven that $WXYZ$ has a larger area than $ABCD$.

Suppose there is a convex quadrilateral $ABCD$ with a perimeter of p. Additionally, BC has a length of a and the lengths of AB, AD, and CD all equal $\frac{p-a}{3}$ ($0<p$, $0<a<\frac{p}{2}$).

To find the angle of $\angle ABC$ when $ABCD$ has the largest area, suppose the length of AC is denoted as b, $\angle ABC$ has an angle of θ, and $\angle ADC$ has an angle of α ($0<b<\frac{2(p-a)}{3}$, $0<\theta<\pi$, $0<\alpha<\pi$).

Using the law of cosines[8], the square of b can be written as shown below:

$$b^2=a^2+\left(\frac{p-a}{3}\right)^2-2a\left(\frac{p-a}{3}\right)\cos\theta$$

Using the equation for the square of b, the equation for the cosine of α can be derived as shown below:

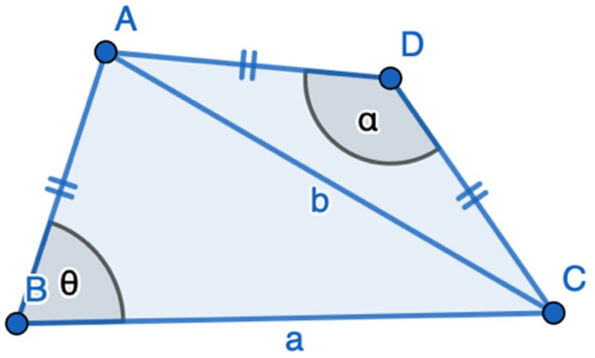

Fig. 11 A visual representation of *ABCD* that includes $\angle ABC$, which has an angle of θ, $\angle ADC$, which has an angle of α, *AC*, the length of *AC*, which is *b*, and the length of *BC*, which is *a*

$$\cos\alpha = \frac{2\left(\frac{p-a}{3}\right)^2 - b^2}{2\left(\frac{p-a}{3}\right)^2} = \frac{2\left(\frac{p-a}{3}\right)^2 - \left(a^2 + \left(\frac{p-a}{3}\right)^2 - 2a\left(\frac{p-a}{3}\right)\cos\theta\right)}{2\left(\frac{p-a}{3}\right)^2}$$
$$= \frac{\left(\frac{p-a}{3}\right)^2 - a^2 + 2a\left(\frac{p-a}{3}\right)\cos\theta}{2\left(\frac{p-a}{3}\right)^2}$$

As the cosine of α can be represented as a function of θ, the formula for the area of *ABCD* can also be written as a function of θ. Let $S(\theta)$ be the function for the area of *ABCD*. $S(\theta)$ can be written as shown below:

$$\begin{aligned}
S(\theta) &= \frac{1}{2}a\left(\frac{p-a}{3}\right)\sin\theta + \frac{1}{2}\left(\frac{p-a}{3}\right)^2\sin\alpha \\
&= \frac{1}{2}a\left(\frac{p-a}{3}\right)\sin\theta + \frac{1}{2}\left(\frac{p-a}{3}\right)^2\sqrt{1-\cos^2\alpha} \\
&= \frac{1}{2}a\left(\frac{p-a}{3}\right)\sin\theta + \frac{1}{2}\left(\frac{p-a}{3}\right)^2\sqrt{1-\left(\frac{\left(\frac{p-a}{3}\right)^2 - a^2 + 2a\left(\frac{p-a}{3}\right)\cos\theta}{2\left(\frac{p-a}{3}\right)^2}\right)^2} \\
&= \frac{1}{2}a\left(\frac{p-a}{3}\right)\sin\theta + \frac{1}{4}\sqrt{\left(2\left(\frac{p-a}{3}\right)^2\right)^2 - \left(\left(\frac{p-a}{3}\right)^2 - a^2 + 2a\left(\frac{p-a}{3}\right)\cos\theta\right)^2} \\
&= \frac{1}{2}a\left(\frac{p-a}{3}\right)\sin\theta + \frac{1}{4}\sqrt{3\left(\frac{p-a}{3}\right)^4 - a^4 - 4a^2\left(\frac{p-a}{3}\right)^2\cos^2\theta + 2a^2\left(\frac{p-a}{3}\right)^2 + 4a^3\left(\frac{p-a}{3}\right)\cos\theta - 4a\left(\frac{p-a}{3}\right)^3\cos\theta} \\
&= \frac{1}{2}a\left(\frac{p-a}{3}\right)\sin\theta + \frac{1}{4}\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\theta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\theta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}
\end{aligned}$$

The derivative of $S(\theta)$ can be written as shown below:

$$\begin{aligned}
S'(\theta) &= \frac{1}{2}a\left(\frac{p-a}{3}\right)\cos\theta \\
&\quad + \frac{8a^2\left(\frac{p-a}{3}\right)^2\sin\theta\cos\theta - 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\sin\theta}{8\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\theta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\theta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}} \\
&= \frac{1}{2}a\left(\frac{p-a}{3}\right)\cos\theta \\
&\quad + \frac{a\left(\frac{p-a}{3}\right)\left(2a\left(\frac{p-a}{3}\right)\cos\theta - \left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\right)\sin\theta}{2\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\theta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\theta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}}
\end{aligned}$$

Suppose the value of θ is β radians when $S'(\theta)$ is equal to zero. For the values of θ that are more than 0 radians and less than π radians, if the value of θ is smaller than β radians, $S'(\theta)$ is positive. However, if the value of θ is greater than β radians, $S'(\theta)$ is negative. As anticipated, when the value of θ is β radians, $S'(\theta)$ is equal to zero. This property of $S'(\theta)$ holds true for all permissible values of the constants p and a.

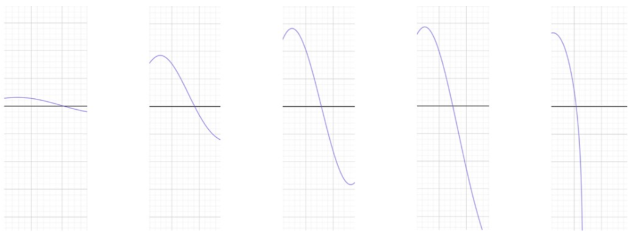

Fig. 12 The graphs of the derivative of the function $S(\theta)$, in which θ is more than 0 radians and less than π radians, for different permissible values of the constants p and a

Therefore, for the values of θ that are more than 0 radians and less than β radians, $S(\theta)$ is strictly increasing. Similarly, for the values of θ that are more than β radians and less than π radians, $S(\theta)$ is strictly decreasing. For this reason, $S(\theta)$ has a maximum value if and only if the value of θ is β radians. As $S(\theta)$ represents the function for the area of *ABCD* for the values of θ that are more than 0 radians and less than π radians, β must be determined to find the value of θ at which *ABCD* has the maximum area. As β is the value of θ when $S'(\theta)$ is equal to zero, $S'(\beta)$ is equal to zero.

$$\begin{aligned}
S'(\beta) &= \frac{1}{2}a\left(\frac{p-a}{3}\right)\cos\beta \\
&\quad + \frac{a\left(\frac{p-a}{3}\right)\left(2a\left(\frac{p-a}{3}\right)\cos\beta - \left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\right)\sin\beta}{2\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\beta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}} = 0
\end{aligned}$$

$$\frac{1}{2}a\left(\frac{p-a}{3}\right)\left(\cos\beta + \frac{\left(2a\left(\frac{p-a}{3}\right)\cos\beta - \left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\right)\sin\beta}{\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\beta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}}\right) = 0$$

$$\cos\beta + \frac{\left(2a\left(\frac{p-a}{3}\right)\cos\beta - \left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\right)\sin\beta}{\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\beta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}} = 0$$

$$\begin{aligned}
&\left(2a\left(\frac{p-a}{3}\right)\cos\beta - \left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\right)\sin\beta \\
&= -\cos\beta\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\beta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4}
\end{aligned}$$

To eliminate the square root from the right side of the equation above, both sides of the equation can be squared and then simplified as shown below:

$$\begin{aligned}
&\left(2a\left(\frac{p-a}{3}\right)\cos\beta - \left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\right)^2\sin^2\beta \\
&= \cos^2\beta\left(-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta + 4a\left(\frac{p-a}{3}\right)\left(a^2 - \left(\frac{p-a}{3}\right)^2\right)\cos\beta - a^4 + 2a^2\left(\frac{p-a}{3}\right)^2 + 3\left(\frac{p-a}{3}\right)^4\right)
\end{aligned}$$

$$\left(2a\left(\frac{p-a}{3}\right)\cos\beta-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)^2\sin^2\beta$$
$$=\cos^2\beta\left(-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta+4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4\right)$$

$$\begin{aligned}&\left(2a\left(\frac{p-a}{3}\right)\cos\beta-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)^2\sin^2\beta\\
&=\left(4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\right)(1-\cos^2\beta)\\
&=4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\\
&\quad-4a^2\left(\frac{p-a}{3}\right)^2\cos^4\beta+4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos^3\beta-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\cos^2\beta\end{aligned}$$

$$\begin{aligned}&=\cos^2\beta\left(-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta+4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4\right)\\
&=-4a^2\left(\frac{p-a}{3}\right)^2\cos^4\beta+4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos^3\beta+\left(-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4\right)\cos^2\beta\end{aligned}$$

$$\begin{aligned}&4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\cos^2\beta\\
&=\left(-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4\right)\cos^2\beta\end{aligned}$$

$$\begin{aligned}&4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\\
&\quad-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\cos^2\beta+\left(a^4-2a^2\left(\frac{p-a}{3}\right)^2-3\left(\frac{p-a}{3}\right)^4\right)\cos^2\beta=0\end{aligned}$$

$$\begin{aligned}&4a^2\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\\
&\quad-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\cos^2\beta+\left(a^4-2a^2\left(\frac{p-a}{3}\right)^2-3\left(\frac{p-a}{3}\right)^4\right)\cos^2\beta\\
&=\left(4a^2\left(\frac{p-a}{3}\right)^2-4\left(\frac{p-a}{3}\right)^4\right)\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\\
&=4\left(\frac{p-a}{3}\right)^2\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos^2\beta-4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)^2\\
&=\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\left(4\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)\\
&=\frac{1}{9}\left(-p^2+2ap+8a^2\right)\left(4\left(\frac{p-a}{3}\right)^2\cos^2\beta-4a\left(\frac{p-a}{3}\right)\cos\beta+\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)\\
&=\frac{1}{9}(4a-p)(2a+p)\left(2\left(\frac{p-a}{3}\right)\cos\beta-\left(a-\left(\frac{p-a}{3}\right)\right)\right)\left(2\left(\frac{p-a}{3}\right)\cos\beta-\left(a+\left(\frac{p-a}{3}\right)\right)\right)\\
&=\frac{1}{9}(4a-p)(2a+p)\left(2\left(\frac{p-a}{3}\right)\cos\beta-\frac{4a-p}{3}\right)\left(2\left(\frac{p-a}{3}\right)\cos\beta-\frac{p+2a}{3}\right)=0\end{aligned}$$

$$(4a-p)\left(2\left(\frac{p-a}{3}\right)\cos\beta-\frac{4a-p}{3}\right)\left(2\left(\frac{p-a}{3}\right)\cos\beta-\frac{p+2a}{3}\right)=0$$

Based on the final equation, $S'(\theta)$ seems to be equal to zero when p is equal to $4a$ or the cosine of θ is equal to $\frac{4a-p}{2(p-a)}$ or $\frac{p+2a}{2(p-a)}$. However, during one of the stages of the simplification process, both sides of the equation were squared. So, all the potential solutions must be verified.

In the case in which the cosine of θ is equal to $\frac{4a-p}{2(p-a)}$, the sine of θ can be written as shown below:

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\frac{4a-p}{2(p-a)}\right)^2}=\frac{\sqrt{4(p-a)^2-(4a-p)^2}}{2(p-a)}=\frac{\sqrt{3p^2-12a^2}}{2(p-a)}$$

As θ is more than 0 radians and less than π radians, the sine of θ is positive[13]. Using the sine of θ, the value of $S'(\theta)$ is shown below:

$$\begin{aligned}S'(\theta)&=\frac{1}{2}a\left(\frac{p-a}{3}\right)\cos\theta+\frac{a\left(\frac{p-a}{3}\right)\left(2a\left(\frac{p-a}{3}\right)\cos\theta-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)\sin\theta}{2\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\cos^2\theta+4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\cos\theta-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4}}\\
&=\frac{1}{2}a\left(\frac{p-a}{3}\right)\frac{4a-p}{2(p-a)}+\frac{a\left(\frac{p-a}{3}\right)\left(2a\left(\frac{p-a}{3}\right)\frac{4a-p}{2(p-a)}-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)\sqrt{\frac{3p^2-12a^2}{2(p-a)}}}{2\sqrt{-4a^2\left(\frac{p-a}{3}\right)^2\left(\frac{4a-p}{2(p-a)}\right)^2+4a\left(\frac{p-a}{3}\right)\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\frac{4a-p}{2(p-a)}-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4}}\\
&=\frac{(4a-p)a}{12}+\frac{a\left(\frac{(4a-p)a}{3}-\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\right)\sqrt{\frac{3p^2-12a^2}{6}}}{2\sqrt{-a^2\left(\frac{4a-p}{3}\right)^2+4a\left(a^2-\left(\frac{p-a}{3}\right)^2\right)\frac{4a-p}{6}-a^4+2a^2\left(\frac{p-a}{3}\right)^2+3\left(\frac{p-a}{3}\right)^4}}\\
&=\frac{(4a-p)a}{12}+\frac{a\left(\frac{12a^2-3ap+p^2-2ap-8a^2}{9}\right)\sqrt{3p^2-12a^2}}{12\sqrt{-\left(a^2\left(\frac{4a-p}{3}\right)^2-a\left(2a^2-\left(\frac{p-a}{3}\right)^2\right)\frac{4a-p}{3}+\left(a^2-3\left(\frac{p-a}{3}\right)^2\right)\left(a^2+\left(\frac{p-a}{3}\right)^2\right)\right)}}\\
&=\frac{(4a-p)a}{12}+\frac{a\left(\frac{p^2-5ap+4a^2}{9}\right)\sqrt{3p^2-12a^2}}{12\sqrt{-\left(a\left(\frac{4a-p}{3}\right)-\left(a^2-3\left(\frac{p-a}{3}\right)^2\right)\right)\left(a\left(\frac{4a-p}{3}\right)-\left(a^2+\left(\frac{p-a}{3}\right)^2\right)\right)}}\\
&=\frac{(4a-p)a}{12}+\frac{a\left(\frac{(p-a)(p-4a)}{9}\right)\sqrt{3p^2-12a^2}}{12\sqrt{-\left(\frac{4a^2-ap+a^2}{3}-a^2+\frac{p^2-2ap+a^2}{3}\right)\left(\frac{4a^2-ap}{3}-a^2-\frac{p^2-2ap+a^2}{9}\right)}}\\
&=\frac{(4a-p)a}{12}+\frac{a\left(\frac{(p-a)(p-4a)}{9}\right)\sqrt{3p^2-12a^2}}{12\sqrt{-\left(\frac{p^2-3ap+2a^2}{3}\right)\left(\frac{-p^2-2ap+2a^2}{9}\right)}}=\frac{(4a-p)a}{12}+\frac{a\left(\frac{(p-a)(p-4a)}{9}\right)\sqrt{3(p-2a)(p+2a)}}{4\sqrt{\frac{(p-a)(p-2a)(p-a)(p+2a)}{3}}}\\
&=\frac{(4a-p)a}{12}+\frac{a(p-a)(p-4a)}{36\sqrt{\frac{(p-a)^2}{9}}}=\frac{a(4a-p)}{12}+\frac{a(p-a)(p-4a)}{12(p-a)}=-\frac{a(p-4a)}{12}+\frac{a(p-4a)}{12}=0\end{aligned}$$

Thus, $S'(\theta)$ is equal to zero when the cosine of θ is equal to $\frac{4a-p}{2(p-a)}$.

In the case in which the cosine of θ is equal to $\frac{p+2a}{2(p-a)}$, the sine of θ can be written as shown below:

$$\begin{aligned}\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\frac{p+2a}{2(p-a)}\right)^2}&=\frac{\sqrt{4(p-a)^2-(p+2a)^2}}{2(p-a)}\\
&=\frac{\sqrt{3p^2-12ap}}{2(p-a)}\end{aligned}$$

As θ is more than 0 radians and less than π radians, the sine of θ is positive[13]. Using the sine of θ, the value of $S'(\theta)$ is shown below:

$$\begin{aligned}S'(\theta)&=\frac{(p+2a)a}{12}+\frac{a\cdot\frac{(p+2a)(p-a)}{9}\cdot\sqrt{3p(p-4a)}}{12\sqrt{\frac{p(p-a)(p-a)(p-4a)}{3}}}\\
&=\frac{(p+2a)a}{12}+\frac{a(p+2a)(p-a)}{12(p-a)}=\frac{(p+2a)a}{12}+\frac{a(p+2a)}{12}=\frac{(p+2a)a}{6}>0\end{aligned}$$

Thus, $S'(\theta)$ is not equal to zero when the cosine of θ is equal to $\frac{p+2a}{2(p-a)}$.

In the case in which p is equal to $4a$, the equation for the value of $S'(\theta)$ is shown below:

$$\begin{aligned}S'(\theta)&=\frac{1}{2}a^2\cos\theta+\frac{a^2(2a^2\cos\theta)\sin\theta}{2\sqrt{-4a^4\cos^2\theta+4a^4}}\\
&=\frac{1}{2}a^2\cos\theta+\frac{a^2(2a^2\cos\theta)\sin\theta}{4a^2\sin\theta}=\frac{1}{2}a^2\cos\theta+\frac{a^2\cos\theta}{2}=a^2\cos\theta\end{aligned}$$

For $S'(\theta)$ to equal zero in the case in which p is equal to $4a$, the cosine of θ must be zero. Coincidentally, in the case in which p is equal to $4a$, $\frac{4a-p}{2(p-a)}$ is equal to zero. So, when p is equal to $4a$, $S'(\theta)$ is equal to zero if and only if the cosine of θ is equal to $\frac{4a-p}{2(p-a)}$.

After evaluating all the potential solutions, $S'(\theta)$ is equal to zero if and only if the cosine of θ is equal to $\frac{4a-p}{2(p-a)}$. So, $ABCD$ has a maximum area if and only if the value of θ is the inverse cosine of $\frac{4a-p}{2(p-a)}$.

Using the equation for the cosine of α, when the value of θ is the inverse cosine of $\frac{4a-p}{2(p-a)}$, the value of the cosine of α is as shown below:

$$\cos\alpha = \frac{\left(\frac{p-a}{3}\right)^2 - a^2 + 2a\left(\frac{p-a}{3}\right)\frac{4a-p}{2(p-a)}}{2\left(\frac{p-a}{3}\right)^2} = 1 - \frac{a^2 + \left(\frac{p-a}{3}\right)^2 - \frac{(4a^2-ap)}{3}}{2\left(\frac{p-a}{3}\right)^2}$$

$$= 1 - \frac{\frac{(p^2+ap-2a^2)}{9}}{2\left(\frac{p-a}{3}\right)^2} = 1 - \frac{(p+2a)}{2(p-a)} = \frac{p-4a}{2(p-a)}$$

Thus, when the value of θ is the inverse cosine of $\frac{4a-p}{2(p-a)}$, the value of the cosine of α is the additive inverse of the cosine of θ. As both θ and α have a value that is more than 0 radians and less than π radians, the value of θ is π radians subtracted by the value of α. So, the sum of the values of θ and α is π radians. Hence, the sum of $\angle ABC$ and $\angle ADC$ is 180.

As the sum of $\angle ABC$ and $\angle ADC$ is 180, $ABCD$ is a cyclic quadrilateral. Let the circumcenter of $ABCD$ be referred to as point E.

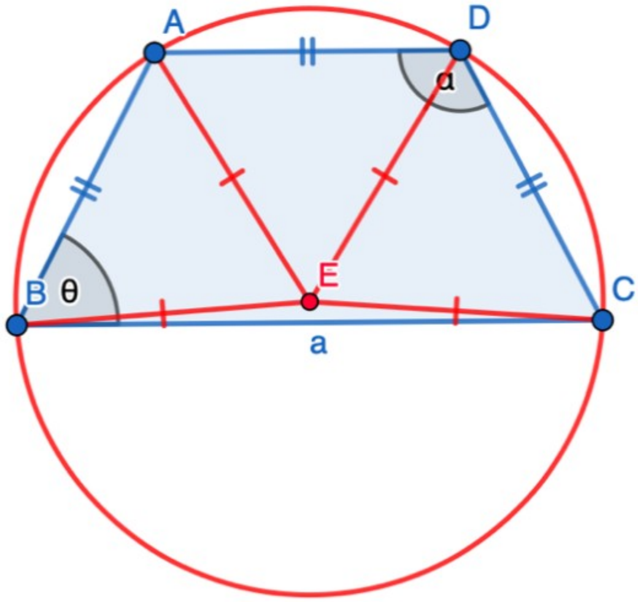

Fig. 13 A visual representation of $ABCD$, which includes $\angle ABC$ (angle θ), $\angle ADC$ (angle α), and the length of BC (a), and a visual representation of the circumcircle of $ABCD$, which includes the circumcenter E and AE, BE, CE, DE

$\triangle EAB$, $\triangle EDA$, and $\triangle ECD$ are all congruent isosceles triangles because AB, CD, and AD all have the same length and AE, BE, CE, and DE all have the same length. Thus, $\angle ABE$, $\angle BAE$, $\angle DAE$, $\angle ADE$, $\angle CDE$, and $\angle DCE$ all have the same angles. As the sum of $\angle BAE$ and $\angle DAE$ is $\angle BAD$ and the sum of $\angle ADE$ and $\angle CDE$ is $\angle ADC$, $\angle BAD$ has the same angle as $\angle ADC$.

In short, when the cosine of θ is $\frac{4a-p}{2(p-a)}$ in which the area of $ABCD$ is at its maximum, $\angle BAD$ and $\angle CDA$ also have the same angles.

Thus, if there is a convex quadrilateral $ABCD$ in which AB, CD, and AD all have the same lengths and $\angle BAD$ and $\angle CDA$ do not have the same angle, there exists a convex quadrilateral with a larger area, denoted $WXYZ$, in which the lengths of WX, YZ, and WZ all equal the length of AB, the length of XY is equal to the length of BC, and $\angle XWZ$ has the same angle as $\angle YZW$, proving Theorem 2.1.

Theorem 2.2 The largest n-sided base polygon is the regular n-sided base polygon.

Theorem 1.2 affirms that the largest n-sided base polygon is an equilateral n-sided base polygon. As the equilateral 3-sided base polygon is the regular 3-sided base polygon, the largest 3-sided base polygon is the regular 3-sided base polygon.

For every convex non-equiangular equilateral 4-sided base polygon, Theorem 2.1 demonstrates that there exists a 4-sided base polygon with a larger area. To prove the existence of a n-sided base polygon with a larger area for every convex non-equiangular equilateral n-sided base polygon with at least 5 sides, let's suppose that there is a convex non-equiangular equilateral n-sided polygon, referred to as polygon $A_1A_2\ldots A_n$, in which $\angle A_iA_{i+1}A_{i+2}$ and $\angle A_{i+1}A_{i+2}A_{i+3}$ are two arbitrary adjacent angles that do not have the same angle ($5 \le n$, $2 \le i \le n-4$).

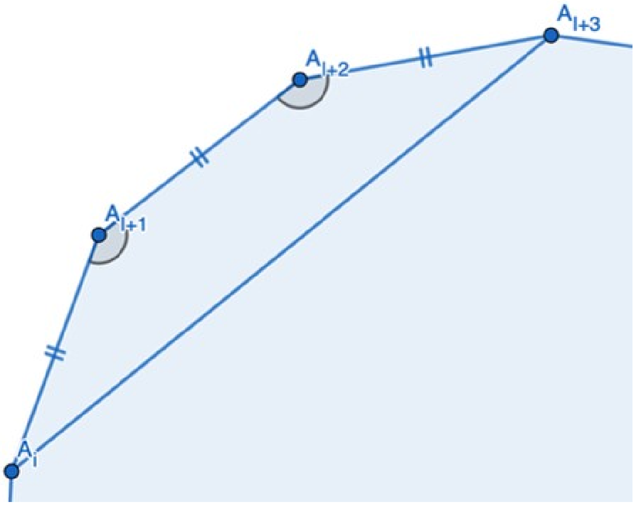

Fig. 14 A visual representation of polygon $A_1A_2\ldots A_n$ that focuses on vertices A_i, A_{i+1}, A_{i+2} and A_{i+3} and includes A_iA_{i+3}, $\angle A_iA_{i+1}A_{i+2}$, and $\angle A_{i+1}A_{i+2}A_{i+3}$

To prove that there exists a n-sided base polygon that has a larger area compared to polygon $A_1A_2\ldots A_n$, polygon $A_1A_2\ldots A_n$ must first be divided into two separate polygons, quadrilateral $A_iA_{i+1}A_{i+2}A_{i+3}$ and polygon $A_1A_2\ldots A_iA_{i+3}\ldots A_n$. Quadrilateral $A_iA_{i+1}A_{i+2}A_{i+3}$ must be convex because $\angle A_iA_{i+1}A_{i+2}$ and $\angle A_{i+1}A_{i+2}A_{i+3}$ are both less than 180 and $\angle A_{i+1}A_iA_{i+3}$ and $\angle A_{i+2}A_{i+3}A_{i+1}$ are also less than 180 because they are smaller than $\angle A_{i+1}A_iA_{i-1}$ and $\angle A_{i+2}A_{i+3}A_{i+4}$, which are both smaller than 180, respectively. As quadrilateral $A_iA_{i+1}A_{i+2}A_{i+3}$ is convex and $\angle A_iA_{i+1}A_{i+2}$ and $\angle A_{i+1}A_{i+2}A_{i+3}$ do not have the same angle, using Theorem 2.1, it can be proven that there exists a convex quadrilateral with a larger area, denoted quadrilateral $ABCD$, in which the lengths of AB, CD, and AD all equal the length of A_iA_{i+1}, the length of BC is equal to the length of A_iA_{i+3}, and $\angle BAD$ has the same angle as $\angle CDA$. A new n-sided polygon can be composed by shifting the position and rotating $ABCD$ so that BC coincides with A_iA_{i+3} and vertex A and vertex D are not on the interior or the boundary of polygon $A_1A_2\ldots A_iA_{i+3}\ldots A_n$. The new n-sided polygon can be referred to as polygon $A_1A_2\ldots A_iADA_{i+3}\ldots A_n$. As $ABCD$ has a larger area compared to $A_iA_{i+1}A_{i+2}A_{i+3}$, polygon $A_1A_2\ldots A_iADA_{i+3}\ldots A_n$ has a larger area compared to polygon $A_1A_2\ldots A_n$. In addition, as the lengths of AB, CD, and AD are all equal to the length of A_iA_{i+1}, polygon $A_1A_2\ldots A_iADA_{i+3}\ldots A_n$ is also a base polygon because it has the same perimeter as polygon $A_1A_2\ldots A_n$.

In short, for every convex non-equiangular equilateral n-sided base polygon with at least 4 sides, there exists a n-sided base polygon with a larger area. Theorem 1.2 states that the largest n-sided base polygon must be a convex equilateral base polygon. Thus, using Theorem 0.1, it can be proven that the largest n-sided base polygon with at least 4 sides is regular. Hence, the largest n-sided base polygon is the regular n-sided base polygon, proving Theorem 2.2.

Regular Polygons and the Circle

Theorem 2.2 affirms that the largest base polygon is a regular polygon. However, to find the largest base polygon, the area of every regular base polygon must be compared. Suppose there is a regular n-sided base polygon $A_1A_2\ldots A_n$. In addition, point O is the center of polygon $A_1A_2\ldots A_n$, $\angle OA_1A_2$ has an angle of θ, and point B is the midpoint of A_1A_2. As $\triangle OA_1A_2$ is an isosceles triangle, $\angle OBA_1$ has an angle of 90 ($3 \leq n$).

The equation for the value of θ in radians can be written as shown below:

$$\theta = \frac{1}{2}\left(\pi - \frac{2\pi}{n}\right) = \frac{\pi}{2} - \frac{\pi}{n}$$

Using the equation for the value of θ, the function for the area of polygon $A_1A_2\ldots A_n$, denoted $S(n)$, can be written as shown below:

$$S(n) = n\left(\frac{1}{2n}\right)^2 \tan\theta = \frac{1}{4n}\tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = \frac{1}{4n\tan\frac{\pi}{n}}$$

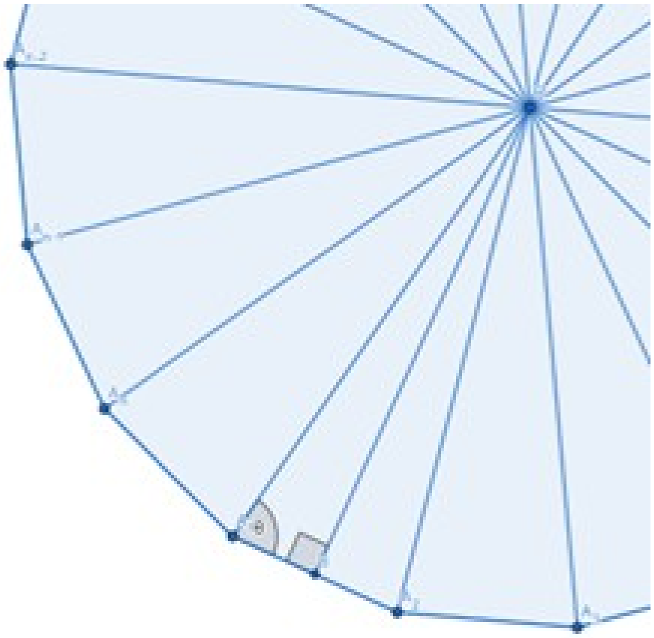

Fig. 15 A visual representation of the regular polygon $A_1A_2\ldots A_n$ that focuses on center O and vertices $A_1, A_2, A_3, A_{n-2}, A_{n-1}$, and A_n and includes $\angle OA_1A_2$ (angle θ), point B, and $\angle OBA_1$ (90)

Let $A(x)$ be defined as shown below:

$$A(x) = \frac{1}{4x\tan\frac{\pi}{x}}$$

The derivative of $A(x)$ can be written as shown below:

$$A'(x) = -\frac{4\tan\frac{\pi}{x} - \frac{4\pi}{x\cos^2\frac{\pi}{x}}}{\left(4x\tan\frac{\pi}{x}\right)^2} = \frac{\frac{\pi}{x\cos^2\frac{\pi}{x}} - \tan\frac{\pi}{x}}{4x^2\tan^2\frac{\pi}{x}}$$

$$= \frac{\pi - x\sin\frac{\pi}{x}\cos\frac{\pi}{x}}{4x\sin^2\frac{\pi}{x}} = \frac{1 - \frac{x}{\pi}\sin\frac{\pi}{x}\cos\frac{\pi}{x}}{\frac{4x}{\pi}\sin^2\frac{\pi}{x}} = \frac{1 - \frac{x}{2\pi}\sin\frac{2\pi}{x}}{\frac{4x}{\pi}\sin^2\frac{\pi}{x}} = \frac{1 - \frac{\sin\frac{2\pi}{x}}{\frac{2\pi}{x}}}{\frac{4x}{\pi}\sin^2\frac{\pi}{x}}$$

The inequality shown below is true for all i greater than 0 and less than $\frac{\pi}{2}$[14]:

$$\frac{\sin i}{i} < \frac{1}{\cos i}$$

This inequality can be rearranged and altered as shown below:

$$\frac{\sin i}{i} < \frac{1}{\cos i}$$
$$\sin i\cos i < i$$
$$2\sin i\cos i < 2i$$
$$\sin 2i < 2i$$

For every value of x in which x is more than or equal to 3, $\frac{\pi}{x}$ is greater than 0 and less than $\frac{\pi}{2}$. Therefore, using the above inequality, it can be proven that $\frac{\sin\frac{2\pi}{x}}{\frac{2\pi}{x}}$ is less than 1 because the sine of $\frac{2\pi}{x}$ is smaller than $\frac{2\pi}{x}$ for every value of x in which x is more than or equal to 3. As a result, $1-\frac{\sin\frac{2\pi}{x}}{\frac{2\pi}{x}}$ has a positive value. In addition, $\frac{x}{\pi}$ and the sine of $\frac{\pi}{x}$ are also positive. Thus, $A'(x)$ is positive for every value of x in which x is more than or equal to 3. So, $A(x)$ is a strictly increasing function for every value of x greater than or equal to 3. So, for every n more than or equal to 3, $A(n+1)$ is more than $A(n)$. As $S(n)$ is equal to $A(n)$, $S(n+1)$ is more than $S(n)$ for every value of n in which n is an integer more than or equal to 3. Therefore, as the number of sides in a regular base polygon increases, the area also increases. This insinuates that the largest base polygon does not exist because the number of sides can increase indefinitely.

In addition, to determine whether $A(x)$ converges or diverges, the limit of $A(x)$ as x approaches infinity must be evaluated, as shown below:

$$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\frac{1}{4x\tan\frac{\pi}{x}} = \lim_{x\to\infty}\frac{\frac{\pi}{x}}{4\pi\sin\frac{\pi}{x}\cos\frac{\pi}{x}} = \lim_{m\to0^+}\frac{m}{4\pi\sin m\cos m}$$

As $\frac{\sin m}{m}$ would approach 1 as m approaches 0[15], $\lim_{m\to0^+}\frac{m}{4\pi\sin m\cos m}$ would equal $\frac{1}{4\pi}$. Hence, $A(x)$ converges to $\frac{1}{4\pi}$ as x approaches infinity.

$$\lim_{x\to\infty} A(x) = \frac{1}{4\pi}$$

Utilizing the Hausdorff metric and perimeter-area convergence[16], it is possible to show that polygon $A_1A_2\ldots A_n$ converges to the shape of a circle as n approaches infinity.

Suppose C is a circle with a perimeter of 1 which has point O as its center. Let $P_n \subset \mathbb{R}^2$ and $C_0 \subset \mathbb{R}^2$ denote the set of all points on polygon $A_1A_2\ldots A_n$ and C, respectively. Then, as n approaches infinity, P_n converges to C_0 in the Hausdorff metric as shown below:

$$\lim_{n\to\infty} d_H(P_n, C_0) = 0$$

For all n, the perimeters satisfy:

$$\mathscr{H}^1(\partial P_n) = 1 = \mathscr{H}^1(\partial C_0)$$

Hence, the perimeter of polygon $A_1A_2\ldots A_n$ and C are both 1. Similarly to how $A(x)$ converges to $\frac{1}{4\pi}$ as x approached infinity, $S(n)$ would also converge to $\frac{1}{4\pi}$ as n approached infinity as $S(n)$ is equal to $A(n)$. Thus, as n approaches infinity, the area of polygon $A_1A_2\ldots A_n$ approaches $\frac{1}{4\pi}$. As the perimeter of C is 1, it would have a radius of $\frac{1}{2\pi}$. This means that the area of C is $\frac{1}{4\pi}$. Therefore, as n approaches infinity, the area of polygon $A_1A_2\ldots A_n$ approaches the area of C.

$$\lim_{n\to\infty}\mathscr{H}^2(P_n) = \frac{1}{4\pi} = \mathscr{H}^2(C_0)$$

Thus, polygon $A_1A_2\ldots A_n$ converges to a circle with a perimeter of 1 as n approaches infinity.

In conclusion, the largest base polygon does not exist. Therefore, the largest polygon with a set perimeter cannot exist because the equivalence class under similarity which contains the largest base polygon cannot exist. Hence, Conjecture 0.0 has been proven false because it stated that there did exist a polygon with the largest area, which is incorrect.

Thus, within a set of every polygon with the same perimeter, there cannot exist a polygon within this set which has the largest area. However, it has also been demonstrated that as the area of these polygons approaches its upper bound, it approaches the shape of a circle with the same perimeter.

Conclusion

In conclusion, many steps were taken to disprove the existence of a convex polygon with a fixed perimeter that has the maximum area. First, a conjecture, which stated there did exist a polygon with a fixed perimeter which also had the maximum area, was formed in order to be disproven. Next, using the scaling of polygons, it was demonstrated that such a polygon had to be similar to the shape of the polygon with a perimeter of 1 that has the largest area. Then, it had to be proven that the polygon with a perimeter of 1 that had the largest area had to also be convex. To determine whether the polygon with a perimeter of 1 that has the largest area is a convex equilateral polygon, it was proven that every convex non-equilateral convex polygon with a perimeter of 1 could not have the maximum area. Similarly, after proving that every convex non-equiangular equilateral polygon with a perimeter of 1 could not have the maximum area, it revealed that this polygon is a regular polygon. Finally, the areas of all of the regular polygons with a perimeter of 1 were compared to find the convex polygon with a perimeter of 1 that has the largest area. However, as the number of sides in the regular polygons increased, its area also continued to increase and approached a value of $\frac{1}{4\pi}$, which is the area of the circle with a perimeter of 1. Thus, the polygon with a fixed perimeter that has the largest possible area does not exist because the area of a polygon can continue to increase as its shape approaches the shape of a circle with the same perimeter, disproving the conjecture.

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