Derivation of Bell's theorem using Non-Local Games

Hasin Shaykat

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Bell's Theorem shows that quantum mechanics is incompatible with classical locality and determinism. This paper uses the recent framework of non-local games to give a new proof of Bell's theorem. Non-local games illuminate how quantum strategies can outperform classical ones at obtaining outcomes impossible in local, deterministic theories. This paper discusses the framework and power of quantum mechanics to transcend classical limits, hence making Bell's theorem of utmost importance in quantum information science.

Introduction

Bell's Theorem revealed a new perspective of the quantum world where the conventional classical assumptions about the nature of reality were questioned[1]. The theorem F shows that quantum mechanics is fundamentally incompatible with classical theories based on two key principles: locality and determinism. Locality, based off Einstein's theory of relativity, asserts that information cannot travel faster than the speed of light, hereby prohibiting instantaneous communication between distant systems[2]. Realism assumes that the physical properties of a system exist with definite values independently of observation or measurement[3].

Under these classical assumptions, Bell's inequality imposes a bound on the correlations of measurement outcomes of two observers, usually referred to as Alice and Bob, who carry out experiments on a shared quantum system without communicating. Quantum mechanics, however, predicts, and experiments confirm, that entangled quantum systems may exhibit correlations that violate Bell's inequality, thereby demonstrating the failure of classical local hidden variable theories[4,5].

While Bell's original formulation employs inequalities to highlight the contrast between classical and quantum predictions, a modern and intuitive interpretation uses non-local games[6]. Non-local games provide an operational framework to examine quantum entanglement, a key phenomenon enabling quantum systems to achieve outcomes beyond classical limits. These games demonstrate how quantum strategies enable Alice and Bob to win with probabilities that are classically out of reach, thereby providing an alternative and more tractable proof of Bell's theorem.

For instance, the CHSH—Clauser-Horne-Shimony-Holt game—has Alice and Bob independently choose their inputs and produce an output depending on their entangled quantum system[7]. Classical strategies are constrained by local and deterministic rules, whereby the maximal success probability is no greater than 75%. However, quantum strategies entangled with one another can make the success probability reach a value of about 85% -thereby establishing that quantum mechanics really does transcend the classical bounds[8].

The importance of Bell's theorem far exceeds the sphere of theoretical physics alone, having become one of the cornerstones of quantum information science with deep implications for quantum cryptography, quantum computing, and experimental tests of quantum mechanics[9,10] . This contribution will help bring more understanding to the ways through which quantum entanglement defies classical intuitions and opens new frontiers in technology and fundamental physics by studying implications of non-local games.

Methods

I have derived Bell's theorem by using non-local games and probabilistic strategies. The main tool used is the Clauser-Horne-Shimony-Holt (CHSH) inequality, which I used to compare the predictions of quantum mechanics with classical physics.

CHSH Inequality: I began with the inequality in terms of expectation values for the measurement outcomes by Alice and Bob. Quantum mechanical setup here assumes measurements on entangled states; the classical hidden variable theories make their predictions based on pre-existing values.

Through the CHSH inequality, we have demonstrated that quantum mechanical correlations can exceed the bounds imposed by classical physics. The CHSH inequality provides a classical limit of 2 on the correlations between Alice and Bob's measurement outcomes, assuming local hidden variable theories. Quantum mechanics, however, predicts that entangled systems can achieve correlations exceeding this limit, with a maximum value of $2\sqrt{2}$. This violation of the classical bound provides a clear distinction between quantum and classical theories and underscores the nonlocal nature of quantum entanglement.

Methodology: Mathematics

The most general strategy used by Alice and Bob is a probabilistic strategy. The information about their outcome is encoded in a probability distribution

$$p(ab \mid xy) \tag{1}$$

(Eq. 1; Bell, 1964)

Here $p(ab \mid xy)$ represents the joint probability that Alice gets outcome (a) and Bob gets outcome (b) when Alice gets question (x) and Bob gets question (y). In strategies where Alice and Bob have no access to shared probabilistic classical resources, this probability can be factored into individual probabilities. However, when Alice and Bob do share probabilistic classical resources, the joint probability cannot be factored into the product of their individual probabilities for their respective results.:

$$p(ab \mid xy) \neq p(a \mid x)p(b \mid y) \tag{2}$$

(Eq. 2; Bell, 1964)

Results a and b are correlated by sharing some classical resource, even if Alice and Bob are noncommunicating. To analyze this further, consider the factorization of probabilities. Let us start with the (1). This equation is in this form due to the rules of conditional probability:

$$p(ab \mid xy) = p(a \mid xyb)p(b \mid xy)$$

$p(a \mid xyb)$ represents the conditional probability of Alice's result given both inputs x and y and Bob's result b. $p(b \mid xy)$ represents the conditional probability of Bob's result given both inputs x and y.

Now there can be a situation where Alice and Bob can coordinate their responses based on shared classical resources, generating correlations in their answers. This correlation can be due to a pre-shared classical resource which has the value of λ (value of the classical randomness they share).

$$p(ab \mid xy\lambda) = p(a \mid xyb\lambda)p(b \mid xy\lambda) \tag{3}$$

(Eq. 3; Bell, 1964)

Here $p(ab \mid xy\lambda)$ represents the probability that Alice gets outcome (a) and Bob gets outcome (b) when Alice gets question (x) and Bob gets question (y) and that they can use some preshared classical randomness which takes on the value λ. λ can take on discrete or continuous values and is described by the probability distribution $p(\lambda)$, where $p(\lambda)$ is the probability distribution of the shared classical randomness λ. Combining this with the probability above, to get the total probability, we need to consider all possible values of this is done by integrating or averaging over the probability distribution $p(\lambda)$.

$$p(ab \mid xy) = \sum_{\lambda} p(\lambda)p(ab \mid xy\lambda) = \sum_{\lambda} p(\lambda)p(a \mid xyb\lambda)p(b \mid xy\lambda) \tag{4}$$

(Eq. 4; Bell, 1964; Bohm, 1952)

We can simplify the expression by assuming two assumptions, locality and determinism.

(1) **Locality assumption**: Bob's answer does not depend on Alice's question.

With the locality assumption, Bob's result should be independent of Alice's measurement setting x if there is no communication or pre-shared information between them.

Therefore, $p(b \mid xy\lambda)$ simplifies to $p(b \mid y\lambda)$:

$$p(b\lambda xy\lambda) = p(b \mid y\lambda) \tag{5}$$

(Eq. 5; Bell, 1964; Bohm, 1952)

Now consider $p(a \mid xyb)$. Similar to Bob's case, Alice's result a should be independent of Bob's input y .

Therefore,

$$p(a \mid xyb\lambda) = p(a \mid xb\lambda) \tag{6}$$

(Eq. 6; Bell, 1964; Mermin, 1990)

This can be simplified further, as Alice's result should only depend on her own input x and the value of the classical resourceλ , not on Bob's result b since they do not share any prior information.

$$p(a \mid xb\lambda) = p(a \mid x\lambda) \tag{7}$$

(Eq. 7; Bell, 1964; Mermin, 1990)

$$p(a \mid xb) = p(a \mid x) \tag{8}$$

(Eq. 8; Bell, 1964)

Substituting this into Eq. (5), we obtain:

$$p(ab \mid xy) = \sum_{\lambda} p(\lambda)p(a \mid x,\lambda)q(b \mid y,\lambda) \tag{9}$$

(Eq. 9; Bell, 1964; Bohm, 1952; Mermin, 1990)

* To avoid confusion on Alice and Bob's probabilities, we changed $p(b \mid y\lambda)$ into $q(b \mid y\lambda)$ for Bob's probability.

We will introduce non-local games by considering two players, Alice and Bob, who, after the start of the game cannot communicate but collaborate on some common task of producing pairs of answers such that some pre-specified rule is satisfied. A referee randomly and independently selects one of two questions for each player, which we will denote x for Alice and y for Bob, and sends these to them. Alice responds with a, Bob with b, following any strategy they may implement.

Mathematically, their strategies depend on shared resources:

Non-Local Games Version of Bells Theorem:

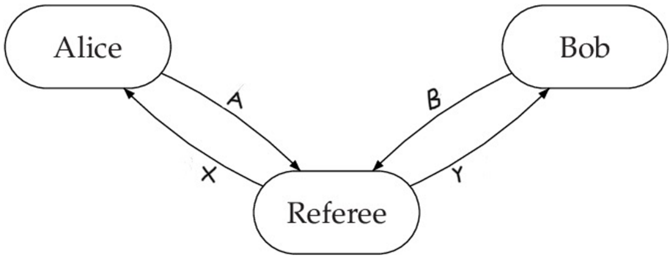


Fig. 1 Caption

1. Classical Resources:
   In the classical case, Alice and Bob may share a hidden variable λ, sampled from some probability distribution $p(\lambda)$, with their responses determined from conditional probabilities $p(a \mid x, \lambda)$ and $p(b \mid y, \lambda)$ such that the joint probability for their outcomes is:

$$p(a,b \mid x,y) = \int_{\lambda} p(\lambda) p(a \mid x, \lambda) p(b \mid y, \lambda) d\lambda. \qquad (10)$$

   These probabilities capture the classical constraint that the correlations between a and b are due only to the shared variable λ .

2. Quantum Resources
   In quantum strategies, Alice and Bob share a quantum state that allows them to achieve correlations beyond the classical limit. Their respective answers are determined by conditional probabilities $p(a \mid x)$ and $p(b \mid y)$, depending on the measurements they perform on the shared quantum state. The joint probability of outcomes a and b in the quantum case can be written as:

$$p(a,b \mid x,y) = \text{function of shared quantum state and local measurement settings.} \qquad (11)$$

3. Question Distributions
   Questions, x for Alice and y for Bob, are independently chosen by the referee according to a prespecified probability distribution p(x, y) . We'll assume that this is uniform, with no bias in the referee's choice of questions. This uniformity is an integral assumption that provides fairness in the game and regularity in the evaluation of strategies that players will employ. The probability p(x,y) is normalized such that:

$$\sum_{x,y} p(x,y) = 1. \qquad (12)$$

4. Measurement Assumptions
   While this paper does not explicitly model measurement operators (e.g., M_x^a and M_y^b), it assumes the functions of the outcomes a and b on the shared resource for the respective inputs x and y. In the quantum case, the measurements on the shared entangled state are presumed to be in correspondence with standard quantum mechanical operations and consistent with the CHSH inequality or similar tests of Bell's theorem.

   In essence, while classical strategies are based on some local hidden variables and the deterministic or probabilistic answers conditioned by λ , quantum strategies can give rise to stronger correlations as a result of entanglement. These stronger correlations manifest themselves in the violation of classical inequalities such as those derived in Bell's theorem.

Table 1:

Rules of the Game 1. Participants: A referee, two players, Alice and Bob, who cannot communicate once the game begins. 2. Separation: Alice and Bob have no way of communicating with each other during the game. This is where the locality assumption is made 3. Questioning: • Each player is asked one question at a time from a predetermined set of questions. The notation for the questions are (X,Y) • The questions are chosen randomly for each player. • The set of possible questions and the rules for answering them are known to both players before the game starts. 4. Answers: Alice and Bob's answers are determined only by their pre-agreed strategy and the question each player receives. Each player must respond with a "yes"or "no"(or 0 or 1). The notation for this is (A,B). 5. Winning Condition: The players win if their answers match the table described below.

The winning condition is based on providing the correct pairs of answers shown below:

Table 2:

X	**Y**	**Winning Condition**
0	0	A=B (Alice and Bob give same answer)
0	1	A=B(Alice and Bob give same answer)
1	0	A=B (Alice and Bob give same answer)
1	1	$A \neq B$ **(Alice and Bob give Oppositive answer)**

Strategies to win the game:

To win the game 75% of the time on average, players can use various classical setups. One possibility is for both players to choose 0 regardless of the question they receive, known as the zero strategy.

$$(x,y)=(0,0)\rightarrow(a,b)=(0,0) \quad : \text{Win}$$
$$(x,y)=(0,1)\rightarrow(a,b)=(0,0) \quad : \text{Win}$$
$$(x,y)=(1,0)\rightarrow(a,b)=(0,0) \quad : \text{Win}$$
$$(x,y)=(1,1)\rightarrow(a,b)=(0,0) \quad : \text{Loss because } A=1-B$$

This notation means that for questions x and y, if they are asked the same question $(0,0)$ or $(1,1)$, they will both answer response 0 for Alice (denoted by a in (a,b)) and Bob (denoted by b in (a,b)). For questions x and y, if they are asked different questions, $(0,1)$ or $(1,0)$, they will still give 0.

Another strategy is the A and 0 strategy. Alice chooses the A strategy where she sends back whatever bit she received, and Bob sends back only 0.

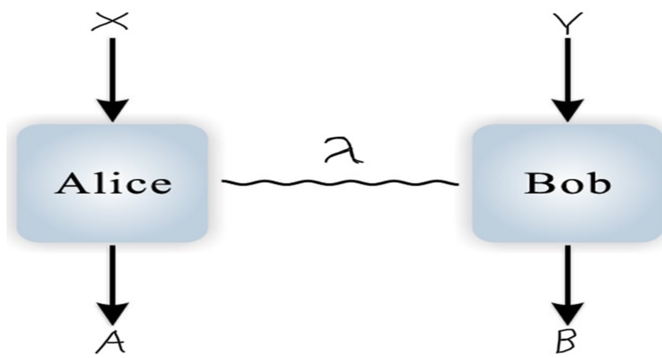
$$(x,y)=(0,0)\rightarrow(a,b)=(0,0) \quad : \text{Win}$$
$$(x,y)=(0,1)\rightarrow(a,b)=(0,0) \quad : \text{Win}$$
$$(x,y)=(1,0)\rightarrow(a,b)=(1,0) \quad : \text{Loss because } A=B$$
$$(x,y)=(1,1)\rightarrow(a,b)=(1,0) \quad : \text{Win}$$

Alice and Bob have a total of 4 deterministic strategies, including the zero strategy of always choosing zero, sending back the received bit, sending back the opposite of the received bit, and always picking 1. Surprisingly, all of these strategies result in a maximal probability of 75% (given that the questions asked are equally likely). But maybe Alice and Bob can use a classical probabilistic strategy. Probabilistic strategies involve using probability to determine the players' answers instead of always returning a fixed response. Alice and Bob can incorporate pre-shared randomness to potentially increase their winning probability. They can use a source of randomness, such as rolling some dice, flipping a coin, or using a random number generator, to decide their answers. We will show next that even using probabilistic strategies only results in a maximal probability of 75%.

Illustration of the non-local game framework: Alice and Bob receive inputs x and y respectively and produce outputs A and B. Communication between them is mediated by the hidden variable λ representing shared information in classical local hidden variable theories.

Classical Winning Probability:

Now to calculate the winning probably of this game we consider our rules again.



Case-by-Case Analysis

Case 1: $X,Y=(0,0)$ The winning condition is $A=B$. Thus, the winning outcomes are $(A,B)=(0,0)$ or $(A,B)=(1,1)$.

$$p_{\text{win}}(00)=p(00\mid 00)+p(11\mid 00) \tag{13}$$

Case 2: $X,Y=(1,0)$ The winning condition is still $A=B$. Thus, the winning outcomes are $(A,B)=(0,0)$ or $(A,B)=(1,1)$.

$$p_{\text{win}}(00)=p(00\mid 10)+p(11\mid 10) \tag{14}$$

Case 3: $X,Y=(0,1)$ The winning condition is still $A=B$. Thus, the winning outcomes are $(A,B)=(0,0)$ or $(A,B)=(1,1)$.

$$p_{\text{win}}(00)=p(00\mid 01)+p(11\mid 01) \tag{15}$$

Case 4: $X,Y=(1,1)$ The winning condition is $A\neq B$. Thus, the winning outcomes are $(A,B)=(0,1)$ or $(A,B)=(1,0)$.

$$p_{\text{win}}(11)=p(01\mid 11)+p(10\mid 11) \tag{16}$$

If the questions are uncorrelated with each other, $P(xy)=p(x)p(y)$ and then overall winning probability is

$$p_{\text{win}}=\sum_{xy}p_{\text{win}}(x,y)p(x)p(y) \tag{17}$$

If each combination is equally likely $p(x)=\frac{1}{2}$ and $p(y)=\frac{1}{2}$, then $P(xy)=\frac{1}{4}$ and the probability becomes

$$p_{\text{win}}=\frac{1}{4}\sum_{xy}p_{\text{win}}(x,y) \tag{18}$$

$$=\frac{1}{4}[p(00\mid 00)+p(11\mid 00)+p(00\mid 01)+p(11\mid 01)+p(00\mid 10)$$
$$+p(11\mid 10)+p(01\mid 11)+p(10\mid 11)]$$

Remember,

$$p(ab \mid xy) = \sum_{\lambda} p(\lambda) p(a \mid x, \lambda) q(b \mid y, \lambda) \tag{19}$$

Using $p(xy) = \frac{1}{4}$ for (x, y) and $p_{\text{win}} = \frac{1}{4}[p_{\text{win}}(0,0) + p_{\text{win}}(1,0) + p_{\text{win}}(0,1) + p_{\text{win}}(1,1)]$

Substituting $p(ab \mid xy)$ into each p_{win} term:

$$p_{\text{win}}(0,0) = \sum_{\lambda} p(\lambda)[p(0 \mid 0,\lambda)q(0 \mid 0,\lambda) + p(1 \mid 0,\lambda)q(1 \mid 0,\lambda)] \tag{20}$$

$$p_{\text{win}}(1,0) = \sum_{\lambda} p(\lambda)[p(0 \mid 1,\lambda)q(0 \mid 0,\lambda) + p(1 \mid 1,\lambda)q(1 \mid 0,\lambda)] \tag{21}$$

$$p_{\text{win}}(0,1) = \sum_{\lambda} p(\lambda)[p(0 \mid 0,\lambda)q(0 \mid 1,\lambda) + p(1 \mid 0,\lambda)q(1 \mid 1,\lambda)] \tag{22}$$

$$p_{\text{win}}(1,1) = \sum_{\lambda} p(\lambda)[p(1 \mid 1,\lambda)q(0 \mid 1,\lambda) + p(0 \mid 1,\lambda)q(1 \mid 1,\lambda)] \tag{23}$$

The overall winning probability then becomes:

$$p_{\text{win}} = \frac{1}{4} \sum_{\lambda} p(\lambda)[p(0 \mid 0,\lambda)q(0 \mid 0,\lambda) + p(1 \mid 0,\lambda)q(1 \mid 0,\lambda)]$$

$$+ \sum_{\lambda} p(\lambda)[p(0 \mid 1,\lambda)q(0 \mid 0,\lambda) + p(1 \mid 1,\lambda)q(1 \mid 0,\lambda)]$$

$$+ \sum_{\lambda} p(\lambda)[p(0 \mid 0,\lambda)q(0 \mid 1,\lambda) + p(1 \mid 0,\lambda)q(1 \mid 1,\lambda)]$$

$$+ \sum_{\lambda} p(\lambda)[p(0 \mid 1,\lambda)q(1 \mid 1,\lambda) + p(1 \mid 1,\lambda)q(0 \mid 1,\lambda)] \tag{24}$$

Factoring out the summations and $p(\lambda)$, the probability becomes:

$$\frac{1}{4} \sum_{\lambda} p(\lambda)[p(0 \mid 0,\lambda)q(0 \mid 0,\lambda) + p(1 \mid 0,\lambda)q(1 \mid 0,\lambda)$$

$$+ p(0 \mid 1,\lambda)q(0 \mid 0,\lambda) + p(1 \mid 1,\lambda)q(1 \mid 0,\lambda) + p(0 \mid 0,\lambda)$$

$$q(0 \mid 1,\lambda) + p(1 \mid 0,\lambda)q(1 \mid 1,\lambda) + p(0 \mid 1,\lambda)q(1 \mid 1,\lambda)$$

$$+ p(1 \mid 1,\lambda)q(0 \mid 1,\lambda)] \tag{25}$$

The expression in the brackets in the above summation can be interpreted as the winning probability given a particular value of classical resource (λ). The overall probability is the averaging of the winning probability of all strategies. (λ contain the information about which strategy Alice and Bob should use).

Remember, for $x = 0$ and $y = 0$,

$$p(0 \mid 0,\lambda) + p(1 \mid 0,\lambda) = 1 \quad \text{and} \quad q(0 \mid 0,\lambda) + q(1 \mid 0,\lambda) = 1$$

Remember, for $x = 1$ and $y = 0$,

$$p(0 \mid 1,\lambda) + p(1 \mid 1,\lambda) = 1 \quad \text{and} \quad q(0 \mid 1,\lambda) + q(1 \mid 1,\lambda) = 1$$

Remember, for $x = 0$ and $y = 1$,

$$p(0 \mid 0,\lambda) + p(1 \mid 0,\lambda) = 1 \quad \text{and} \quad q(0 \mid 0,\lambda) + q(1 \mid 0,\lambda) = 1$$

Remember, for $x = 1$ and $y = 1$,

$$p(0 \mid 1,\lambda) + p(1 \mid 1,\lambda) = 1 \quad \text{and} \quad q(0 \mid 1,\lambda) + q(1 \mid 1,\lambda) = 1$$

(2) Deterministic assumption: Alice and Bob share some pre-shared classical randomness. After knowing the outcome of the classical randomness and their questions, Alice and Bob follow a deterministic strategy, so the probabilities become functions. So $p(a \mid x,\lambda)$ can either be 0 or 1. The same would apply for $q(b \mid y,\lambda)$.

Now we consider all the different cases that may arise.

Case #1: By the sums above, we can choose the value of $p(0 \mid 0,\lambda)$, for example, 1, but then $p(1 \mid 0,\lambda)$ would be 0. We can do this for all the probabilities and calculate using the overall winning probability formula. Then we get:

Table 3: Probabilities and Winning Probability for Case #1

p(0|0,λ)	**p(1|0,λ)**	**q(0|0,λ)**	**q(1|0,λ)**	**p(0|1,λ)**	**p(1|1,λ)**	**q(0|1,λ)**	**q(1|1,λ)**
1	0	1	0	1	0	1	0

Plugging this into the formula we get

$$p_{\text{win}} = \frac{1}{4}[1(1) + 0(0) + 1(1) + 0(1) + 1(1) + 0(0) + 1(0) + 0(1)] = \frac{3}{4}$$

We can choose a different casework and see the resulting probability.

Table 4: Probabilities and Winning Probability for Case #2

p(0|0,λ)	**p(1|0,λ)**	**q(0|0,λ)**	**q(1|0,λ)**	**p(0|1,λ)**	**p(1|1,λ)**	**q(0|1,λ)**	**q(1|1,λ)**
1	0	1	0	1	0	0	1

$$p_{\text{win}} = \frac{1}{4}[1(1) + 0(0) + 1(0) + 0(1) + 0(1) + 1(0) + 0(0) + 0(1)] = \frac{1}{4}$$

In classical strategies, the outputs for Alice and Bob, p(0|x,λ) and q(0|y,λ) are deterministic functions of their inputs and the hidden variable λ. This means that for any input x or y, the output probabilities can only be 0 or 1. Therefore, each input-output pair (e.g., p(0|0,λ) , q(1|1,λ)) has exactly 2 possibilities: the output is either 0 or 1. Considering all input-output pairs for both Alice and Bob:

We can do this for 16 cases and get the result below:

Case #	**p(0|0,λ)**	**p(1|0,λ)**	**q(0|0,λ)**	**q(1|0,λ)**	**p(0|1,λ)**	**p(1|1,λ)**	**q(0|1,λ)**	**q(1|1,λ)**
1	1	0	1	0	1	0	1	0
2	1	0	1	0	1	0	0	1
3	1	0	1	0	0	1	1	0
4	1	0	1	0	0	1	0	1
5	1	0	0	1	1	0	1	0
6	1	0	0	1	1	0	0	1
7	1	0	0	1	0	1	1	0
8	1	0	0	1	0	1	0	1
9	0	1	1	0	1	0	1	0
10	0	1	1	0	1	0	0	1
11	0	1	1	0	0	1	1	0
12	0	1	1	0	0	1	0	1
13	0	1	0	1	1	0	1	0
14	0	1	0	1	1	0	0	1
15	0	1	0	1	0	1	1	0
16	0	1	0	1	0	1	0	1

Case 1:	**Case 2:**	**Case 3:**	**Case 4:**	**Case 5:**	**Case 6:**	**Case 7:**	**Case 8:**
$P_w = \frac{3}{4}$	$P_w = \frac{3}{4}$	$P_w = \frac{1}{4}$	$P_w = \frac{1}{4}$	$P_w = \frac{3}{4}$	$P_w = \frac{3}{4}$	$P_w = \frac{1}{4}$	$P_w = \frac{1}{4}$
Case 9:	**Case 10:**	**Case 11:**	**Case 12:**	**Case 13:**	**Case 14:**	**Case 15:**	**Case 16:**
$P_w = \frac{3}{4}$	$P_w = \frac{3}{4}$	$P_w = \frac{1}{4}$	$P_w = \frac{1}{4}$	$P_w = \frac{3}{4}$	$P_w = \frac{3}{4}$	$P_w = \frac{1}{4}$	$P_w = \frac{1}{4}$

- Alice has 2 inputs (x=0,1) and 2 possible outputs for each input (p(0|x,λ)) resulting in $2^2 = 4$ combinations.

- Bob similarly 2^2 = 4 combinations for his inputs and outputs (q(0| y,λ)).

Thus, the total number of deterministic strategies for Alice and Bob combined is $4 \times 4 = 16$ These 16 cases represent every possible assignment of deterministic strategies for Alice and Bob, ensuring the analysis is exhaustive. By considering all 16 cases and calculating their respective winning probabilities, we establish that the maximum probability of winning is $\frac{3}{4}$. This is because the averaging winning probability of numbers between $\frac{1}{4}$ and $\frac{3}{4}$ can be no bigger than $\frac{3}{4}$. Alice and Bob can win 75% of the rounds in the game if they use optimal classical strategies. This value shows the limitations of classical strategies. An interesting corollary to this result is also to observe that the least probability Alice and Bob can win the game is $\frac{1}{4}$.

Quantum Case Comparison

Step-by-Step Proof

1. Setup and Definitions The CHSH game involves two players, Alice and Bob, who receive random inputs $x, y \in \{0,1\}$ and produce outputs $a, b \in \{0,1\}$ respectively. Their goal is to maximize the winning condition:

$$a \oplus b = x \wedge y \tag{25}$$

where $\oplus$ is addition modulo 2.

In classical strategies, the best they can achieve is a 75% success probability. Quantum strategies exceed this using shared entangled states.

Quantum Strategy Description

Alice and Bob share a maximally entangled Bell state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{26}$$

They perform quantum measurements on their respective qubits based on their inputs x and y:

- Alice uses measurement operators corresponding to angles $\theta_x = x\frac{\pi}{4}$.

- Bob uses measurement operators corresponding to angles $\phi_y = y\frac{\pi}{4} + \frac{\pi}{8}$.

These angles are derived to optimize correlations between Alice's and Bob's outputs under quantum mechanics.

Winning Probability Calculation

The quantum winning probability is calculated using the expectation value of the measurement outcomes:

$$P_{\text{win}} = \frac{1}{4}\sum_{x,y} P(a \oplus b = x \wedge y \mid x,y). \tag{27}$$

For the Bell state and chosen measurement operators, the probabilities are governed by quantum correlations:

$$\langle A_x B_y \rangle = \cos(\phi_y - \theta_x), \tag{28}$$

where $\langle A_x B_y \rangle$ is the expectation value of the measurement outcomes.

For each input pair (x,y):

- When $x \wedge y = 0$ (i.e., $x = 0$ or $y = 0$), the winning condition is $a \oplus b = 0$, and the quantum success probability is:

$$P(a \oplus b = 0 \mid x,y) = \frac{1 + \langle A_x, B_y \rangle}{2}. \tag{29}$$

- When $x \wedge y = 1$ (i.e., $x = 1$ and $y = 1$), the winning condition is $a \oplus b = 1$, and the quantum success probability is:

$$P(a \oplus b = 1 \mid x,y) = \frac{1 - \langle A_x, B_y \rangle}{2}. \tag{30}$$

Expected Correlation Values

For the chosen measurement settings:

$$\text{If } x = 0, y = 0: \quad \phi_0 - \theta_0 = \frac{\pi}{8}, \quad \langle A_0 B_0 \rangle = \cos\left(\frac{\pi}{8}\right). \tag{31}$$

$$\text{If } x = 0, y = 1: \quad \phi_1 - \theta_0 = \frac{3\pi}{8}, \quad \langle A_0 B_1 \rangle = \cos\left(\frac{3\pi}{8}\right). \tag{32}$$

$$\text{If } x = 1, y = 0: \quad \phi_0 - \theta_1 = -\frac{\pi}{8}, \quad \langle A_1 B_0 \rangle = \cos\left(-\frac{\pi}{8}\right). \tag{33}$$

$$\text{If } x = 1, y = 1: \quad \phi_1 - \theta_1 = \frac{\pi}{8}, \quad \langle A_1 B_1 \rangle = \cos\left(\frac{\pi}{8}\right). \tag{34}$$

Combining Probabilities

Using the formula for P_{win}:

$$P_{\text{win}} = \frac{1}{4}\left(3 \cdot \frac{1 + \cos(\pi/8)}{2} + \frac{1 - \cos(3\pi/8)}{2}\right). \tag{35}$$

Simplify:

$$P_{\text{win}} = \frac{1}{4}\left(\frac{3}{2}(1 + \cos(\pi/8)) + \frac{1}{2}(1 - \cos(3\pi/8))\right). \tag{36}$$

Since $\cos(\pi/8) \approx 0.9239$ and $\cos(3\pi/8) \approx 0.3827$, substituting the values, we get:

$$P_{\text{win}} = \frac{1}{4}\left(\frac{3}{2}(1 + 0.9239) + \frac{1}{2}(1 - 0.3827)\right). \tag{37}$$

$$P_{\text{win}} = \frac{1}{4}\left(\frac{3}{2} \cdot 1.9239 + \frac{1}{2} \cdot 0.6173\right). \tag{38}$$

$$P_{\text{win}} = \frac{1}{4}(2.88585 + 0.30865). \tag{39}$$

$$P_{\text{win}} \approx \frac{1}{4} \times 3.1945 = 0.8536. \tag{40}$$

Conclusion

The quantum strategy achieves a success probability of 85.4%, surpassing the classical limit of 75%. This is possible because entanglement allows stronger correlations than classical physics permits, bounded by the Tsirelson limit.

Discussion

The calculations above show that Alice and Bob can win the CHSH game with a maximum probability of 3/4 (75%) using classical resources. But can we do better with quantum resources? The answer is yes: the principles of quantum mechanics allow for an elevated success rate significantly greater than that available classically. More precisely, the success probability of Alice and Bob goes up to approximately 85.4% if they share an entangled quantum state [11], [8]. Entanglement enables the measurement outcomes of two distant particles to be strongly correlated, even when separated by large distances [2], [12] . This phenomenon is not only a cornerstone of quantum mechanics but also forms the foundation of the quantum advantage in the CHSH game. The theoretical derivation of the quantum success probability is based on measurements performed on a shared Bell state [1]. These measurements, based on optimal angles for the CHSH game's winning condition, manifest the nonlocal correlations of quantum mechanics [13].

To relate this theoretical background to the experimental observations, we turn now to the foundational experiments carried out by Alain Aspect and his collaborators in the early 1980s [4], [14]. Aspect's experiments used entangled photons emitted by calcium atoms and rapidly switching polarization analyzer settings such that no classical information exchange between particles could explain the results. The observed violation of Bell's inequalities in these experiments agrees directly with the theoretical predictions derived for the CHSH game [5]. In particular, the experiments demonstrate the maximum violation of the CHSH inequality, corresponding to the quantum success probability of 85.4%.

The game-theoretic formulation of Bell's theorem provides an intuitive way to quantify this violation[6]. The classical limit of the CHSH game success probability, 3/4, translates directly to the Bell inequality constraint[7]. Any experimental observation above this bound is a sign of quantum correlations. In the Aspect experiments, success probabilities were always found to be very near the quantum mechanical prediction, and this validated the theoretical bounds obtained from the CHSH game[15,16].

Interestingly, while quantum strategies increase the probability of winning, they also increase the minimal probability of losing to 14.6%. This is a consequence of the fact that quantum mechanics is based on probabilities, and this basic characteristic distinguishes it from the deterministic theories of classical physics[17,18]

The consequences of these results go far beyond the scope of theoretical physics. Experimental violations of Bell's inequalities have established quantum mechanics as nonlocal, disproving any local hidden variable theory[13]. Moreover, strong experimental evidence of entanglement opened up the way to achievements in quantum technologies, including quantum computing and quantum cryptography[19,10]. Most vividly, it was marked by the recognition of the pioneering work by Alain Aspect, John F. Clauser, and Anton Zeilinger with the award of the 2022 Nobel Prize in Physics[20].

In summary, the CHSH game offers a thrilling framework for connecting theoretical predictions with experimental validations. The success of quantum strategies in the game underlines the special features of quantum mechanics and exemplifies the deep interplay between theory, experiment, and technological innovation.

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