Optimization Unveiled: Exploring Calculus of Variations for Geodesics and Applications in Mathematics, Physics, and Finance

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Oftentimes, situations will come up in which a quantity is to be optimized for the maximum benefit. In this paper, we will use techniques from ordinary differential equations and other parts of calculus to solve problems relating to the calculus of variations, a field of study that is closely related to optimization. We will first derive the fundamental equation used in the calculus of variations: the Euler-Lagrange Equation. We will also demonstrate how the Euler-Lagrange Equation can be used to determine the shapes of geodesics on classical three-dimensional figures, such as cylinders, cones, and spheres, which is important for determining the shortest path between 2 points on the surface of a curve. We will also demonstrate how the calculus of variations can be used in finance and physics by showing how the Euler-Lagrange Equation can be used with certain profit trends to determine an optimal production function and by showing how the Euler-Lagrange Equation can be used to determine the path of objects subject to common physical conditions. In doing this, we show how the Calculus of Variations, specifically the Euler-Lagrange Equation, can be used to find optimal curves for situations involving an integral with a first-order differential equation as the integrand.

Introduction

Optimization questions have always been of interest in the field of mathematics. These types of problems typically deal with the idea that we wish to get the most out of a finite amount of input. For example, analysis and expansion upon knowledge of inequalities, such as the Cauchy-Schwarz inequality and subsequent results that came about relating to it (that are still being researched to this day), shows the long-held interest of mathematicians to optimize certain quantities, like an equation.

The study of the Calculus of Variations centers around the same idea. Problems relating to the Calculus of Variations usually involve a fixed parameter and a quantity that must be optimized. For example, the question of the shortest path between two points in a plane involves the use of the Calculus of Variations. The Calculus of Variations has been an important area of analysis with regard to both pure mathematics and other subjects, such as physics, leading to the proof of many important principles. For example, according to Simmons[1] the use of Calculus of Variations was central to Einstein's study of relativity, as well as Schrödinger's derivation of the wave equation, important for the study of quantum mechanics.

Further reading of Simmons[1], suggests the Calculus of Variations became a topic of interest in analysis when Euler and Lagrange derived what is now known as the Euler-Lagrange Equation. Before then, problems relating to the Calculus of Variations were considered by the ancient Greeks. After the creation of Calculus by Newton and Leibniz, these problems were investigated more deeply, and some of these problems were solved through other techniques. However, after the derivation of the Euler-Lagrange Equation, the Calculus of Variations was established as its own subject, and since then many mathematicians have studied and continue to make breakthroughs in this area of study.

In this paper, we shall prove the fundamental equation for the Calculus of Variations: The Euler-Lagrange Equation. Furthermore, we shall look at many situations covered from both Simmons[1] and Depurkar[2], relating to pure math or otherwise, where principles in the Calculus of Variations will be of particular use. Additionally, we will examine situations that are natural extensions of those mentioned in Simmons[1] and Depurkar[2], and use the Calculus of Variations to gain insights.

Euler-Lagrange Equation

Deriving the Euler-Lagrange Equation

Here, we shall establish some definitions and assumptions that will be used throughout the paper. First, we shall assume that all optimizing functions, $f(x,y,y')$, have continuous partial derivatives with respect to all variables (in our case, those variables are $x, y,$ and y', where x is the independent variable and $y = y(x)$ is the function we are trying to find that optimizes $f(x,y,y')$. Next, define admissible functions, $y(x)$, as ones that satisfy all initial conditions and have continuous second derivatives.

We wish to find an admissible function such that the integral:

$$\int_{x_1}^{x_2} f(x,y,y')\,dx$$

is minimized, where x_1 and x_2 are arbitrary bounds. First, we let $y(x)$ (abbreviated here as y) be the function that actually minimizes the value of the integral. Then, we declare $\underline{y}$ to be any function of the form:

$$\underline{y} = y + \alpha v$$

where v is the variation function of $\underline{y}$ from y, and α is an arbitrary parameter. Note that at $x = x_1$ and $x = x_2$, $v = 0$. This equation also implies that:

$$\underline{y}' = y' + \alpha v'$$

We know that at $\alpha = 0$, the integral must have a minimum value, meaning that if we declare the integral as a function of α and take the derivative with respect to α, the value of the derivative with respect to α at $\alpha = 0$ is 0, so we consider the following integral:

$$\int_{x_1}^{x_2} f(x, \underline{y}, \underline{y}')\, dx$$

Differentiating under the integral sign with respect to α, we have that:

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha}[f(x, \underline{y}, \underline{y}')]\, dx$$

which, after the use of the Multivariable Chain Rule, becomes:

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \underline{y}}\frac{\partial \underline{y}}{\partial \alpha} + \frac{\partial f}{\partial \underline{y}'}\frac{\partial \underline{y}'}{\partial \alpha}\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \underline{y}} v + \frac{\partial f}{\partial \underline{y}'} v'\, dx$$

Plugging in $\alpha = 0$ and setting $I'(0) = 0$, we get:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y} v + \frac{\partial f}{\partial y'} v'\, dx = 0$$

Note that $\underline{y} = y$ and $\underline{y}' = y'$ at $\alpha = 0$, which is how we justify the substitution made. We then split the integral into two and use integration by parts on the integral $\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} v'\, dx$, turning our equation into:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y} v\, dx + v\frac{\partial f}{\partial y'}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] v\, dx = 0$$

Since $v = 0$ at $x = x_1$ and $x = x_2$, we have that:

$$\int_{x_1}^{x_2} v\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\right) dx = 0$$

Our integral must be equal to 0, regardless of what the function v actually is, therefore we can conclude that all functions y that could possibly minimize the integral must satisfy the equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$

which is the well-known Euler-Lagrange Equation, one of the fundamental equations in the Calculus of Variations. In the demonstrations that follow, we will show how the Euler-Lagrange equations can be used to find optimal curves given an integral with a first order differential equation as the integrand. We will show how this equation can be used to show that some curves optimize quantities, such as length, surface area, profit, and time (which we will address in this paper), as well as find the said optimizing curves between points in both two-dimensions and three-dimensions.

Shortest Curve Between Two Points

The Euler-Lagrange Equation can be applied to many problems in the realm of mathematics. For example, Simmons[1] proposes that we find the curve that minimizes the distance between two points in the xy plane, (x_1, y_1) and (x_2, y_2). Here, we consider the formula for arc length for a function $y(x)$, given by:

$$\int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx$$

Let $f(x, y, y') = \sqrt{1 + (y'(x))^2}$. Applying the Euler-Lagrange Equation to f, we have that:

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + (y'(x))^2}}\right] = 0$$

Integrating both sides with respect to x, we have that:

$$\frac{y'}{\sqrt{1 + (y'(x))^2}} = C_1$$

for some arbitrary constant C_1. We then solve for $y'(x)$, which is given by:

$$(y'(x))^2 = \frac{C_1^2}{1 - C_1^2} \quad \Rightarrow \quad y'(x) = \pm\sqrt{\frac{C_1^2}{1 - C_1^2}}$$

The right-hand side of our equation is simply a constant, so let $C_2 = \pm\sqrt{\frac{C_1^2}{1 - C_1^2}}$, which means that $y = C_2 x + C_3$ for some arbitrary constant C_3. $y = C_2 x + C_3$ is, indeed, a straight line, as desired.

Minimized Surface Area Created By a Curve

Simmons[1] proposes the following: we consider two points, (x_1, y_1) and (x_2, y_2), and look for the curve connecting the two points such that when the curve is rotated about the x-axis, the surface area of the resultant shape is minimized. We will assume here that the points are chosen in a way such that a minimizing curve exists. To find this curve, which we will call y, we shall consider the formula for surface area in this case, given by the integral:

$$\int_{x_1}^{x_2} 2\pi y\sqrt{1+(y')^2}\,dx$$

Applying the Euler-Lagrange Equation to the function $F(x,y,y') = 2\pi y\sqrt{1+(y')^2}$, we get that y must satisfy the equation:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

We multiply both sides by y', which gets us:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}y'\right) - \frac{\partial F}{\partial y}y' = 0$$

Since $\frac{\partial f}{\partial x} = 0$, we get that:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}y'\right) - \frac{\partial F}{\partial y}y' = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}y'\right) - \frac{\partial F}{\partial y}y' - \frac{\partial F}{\partial x}$$

$$= \frac{d}{dx}\left(\frac{\partial F}{\partial y}y' - F\right) = 0$$

Integrating both sides and plugging in F, we get:

$$\frac{2\pi y(y')^2}{\sqrt{1+(y')^2}} - 2\pi y\sqrt{1+(y')^2} = c_1 \quad \Rightarrow \quad -\frac{2\pi y}{\sqrt{1+(y')^2}} = c_1$$

Let $c_2 = -\frac{c_1}{2\pi}$. Solving for y', we get that:

$$y' = \frac{\sqrt{y^2-c_2^2}}{c_2} \quad \Rightarrow \quad \frac{c_2 y'}{\sqrt{y^2-c_2^2}} = 1$$

Integrating both sides, we have that:

$$c_2\left(\frac{y}{c_2}\right) = x + c_3$$

where c_3 is an arbitrary constant. This can be manipulated to get that:

$$y = c_2\cosh\left(\frac{x+c_3}{c_2}\right)$$

Minimized Area Under a Curve

Simmons[1] proposes the following: we consider two points, $(0,0)$ and $(1,0)$, and the first-quadrant curve connecting them, y. Suppose that the area under the curve is held constant. Our goal is to find the curve that minimizes the arc length between the two points. To find the curve, we want to find y that minimizes:

$$L = \int_0^1 \sqrt{1+(y')^2}\,dx$$

given that:

$$\int_0^1 y\,dx = A$$

where A is a constant. Here, we will employ the method of Lagrange Multipliers by declaring a function:

$$f(y,y',\lambda) = \sqrt{1+(y')^2} + \lambda y$$

Applying the Euler-Lagrange equation, we see that all possible minimizing functions $f(x)$ must satisfy the equation:

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) - \lambda = 0$$

Adding λ to both sides and integrating, we get

$$\frac{y'}{\sqrt{1+(y')^2}} = \lambda x + c_1$$

Squaring both sides and solving for y', we get that:

$$y' = \frac{\lambda x + c_1}{\sqrt{1-(\lambda x + c_1)^2}}$$

Integrating both sides, we get

$$\lambda(y+c_2) = -\sqrt{1-(\lambda x+c_1)^2} \Rightarrow y + c_2 = -\sqrt{\frac{1}{\lambda^2} - \left(x + \frac{c_1}{\lambda}\right)^2}$$

which is a semi-circular arc of radius $\frac{1}{\lambda}$.

Geodesics on Two Dimensional Surfaces

Shortest Path Along a Cylinder

Consider a cylinder of the form $x^2 + z^2 - r^2 = 0$ in three-dimensional space. Suppose that we choose two points on this figure, and we desire to model the path of the shortest length along the cylinder between these two points, which we will call (x_0,y_0,z_0) at t_0 and (x_1,y_1,z_1) at t_1. This is known as a geodesic, or a curve representing the shortest path between two points on a surface. To find this, we consider the equation for 3-dimensional arc length:

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Now, we let $x = r\cos(\theta)$ and $z = r\sin(\theta)$, which transforms our integral into:

$$\int_{t_0}^{t_1} \sqrt{\left(r\cos(\theta)\frac{d\theta}{dt}\right)^2 + \left(r\sin(\theta)\frac{d\theta}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Simplifying the integral using Pythagorean Identities, we get that our integral becomes:

$$\int_{t_0}^{t_1} \sqrt{r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{y_0}^{y_1} \sqrt{r^2\left(\frac{d\theta}{dy}\right)^2 + 1}\, dy$$

Now, we declare a new function f:

$$f(y,\theta,\theta') = \sqrt{r^2(\theta')^2+1}$$

and apply the Euler-Lagrange Equation:

$$\frac{d}{dy}\left(\frac{\partial f}{\partial \theta'}\right) = \frac{\partial f}{\partial \theta} \quad \Rightarrow \quad \frac{d}{dy}\left[\frac{r^2\theta'}{\sqrt{r^2(\theta')^2+1}}\right] = 0$$

$$\Rightarrow \quad \frac{r^2\theta'}{\sqrt{r^2(\theta')^2+1}} = C \quad \Rightarrow \quad \theta' = \frac{C}{r\sqrt{r^2-C^2}}$$

Note that C is a constant, meaning that the right-hand side of our equation is also a constant, which we will set equal to c. By integrating both sides, we get that:

$$\theta = cy + k$$

where k is a constant. Plugging in our initial conditions and letting θ_0 and θ_1 be our angles at our positions at time t_0 and t_1, respectively, we get that:

$$\theta_0 = \arccos(x_0/r) = cy_0 + k$$

$$\theta_1 = \arccos(x_1/r) = cy_1 + k$$

Solving this system of equations, we get that:

$$c = \frac{\arccos(x_0/r) - \arccos(x_1/r)}{y_0 - y_1}$$

$$k = -\frac{y_1\arccos(x_0/r) + y_0\arccos(x_1/r)}{y_0 - y_1}$$

Therefore, our optimal curve is:

$$\theta = \frac{\arccos\left(\frac{x_0}{r}\right) - \left(\arccos\left(\frac{x_1}{r}\right)\right)}{y_0 - y_1}y + \frac{-y_1\arccos\left(\frac{x_0}{r}\right) + y_0\arccos\left(\frac{x_1}{r}\right)}{y_0 - y_1}$$

Obviously, this tactic of putting x and z in terms of polar coordinates doesn't always work for other objects, and if we modelled this curve between two points on some other three-dimensional shape, it is unlikely that the curve would be the shortest path between said points. However, the general idea of trying to reduce the number of variables using alternative coordinate systems, like polar coordinates, will be useful as we consider geodesics on other objects, such as cones and cylinders.

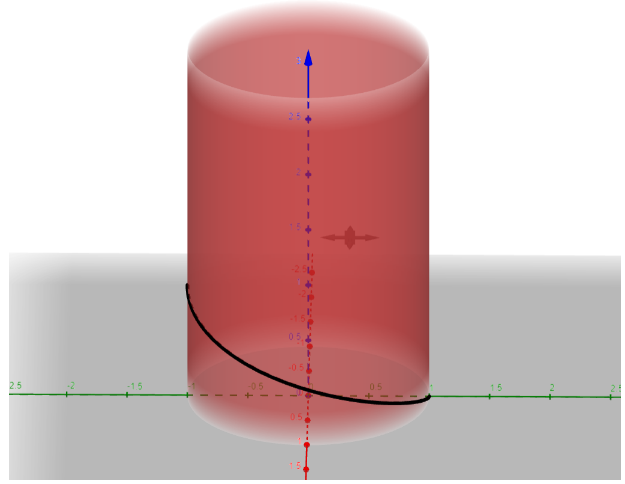

Fig. 1 Graph of the geodesic along the surface of the cylinder

Geodesics on a Cone

Now, consider a cone of the form $x^2 + y^2 = z^2$ with $z \geq 0$. We desire to find the optimal path between two points along the cone, calling them (x_0, y_0, z_0) at t_0 and (x_1, y_1, z_1) at t_1. Again, we will consider the arc-length formula in three dimensions:

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

We let $x = z\cos(\theta) \Rightarrow \frac{dx}{dt} = \frac{dz}{dt}\cos(\theta) - z\sin(\theta)\frac{d\theta}{dt}$ and $y = z\sin(\theta) \Rightarrow \frac{dy}{dt} = \frac{dz}{dt}\sin(\theta) + z\cos(\theta)\frac{d\theta}{dt}$. Our integral becomes:

$$\int_{t_0}^{t_1} \sqrt{2\left(\frac{dz}{dt}\right)^2 + z^2\left(\frac{d\theta}{dt}\right)^2}\, dt = \int_{\theta_0}^{\theta_1} \sqrt{2\left(\frac{dz}{d\theta}\right)^2 + z^2}\, d\theta$$

We can now declare a function f:

$$f(\theta, z, z') = \sqrt{2(z')^2 + z^2}$$

and apply the Euler-Lagrange equation:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right) = \frac{\partial f}{\partial z}$$

We multiply both sides by z' to get:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right)z' - \frac{\partial f}{\partial z}z' = 0$$

Since $\frac{\partial f}{\partial \theta} = 0$, we have that our equation becomes:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right)z' - \frac{\partial f}{\partial z}z' - \frac{\partial f}{\partial \theta} = 0$$

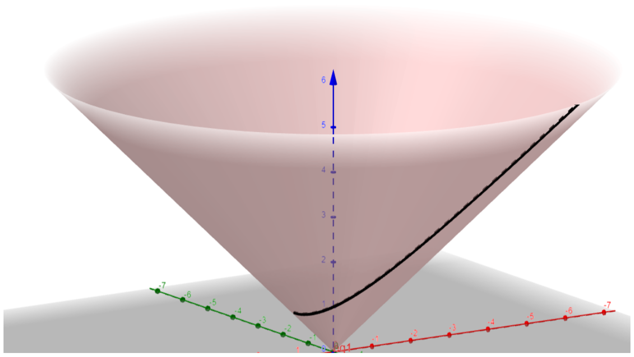

Fig. 2 Graph of the geodesic along the surface of the cone

Integrating both sides, we get:

$$\frac{\partial f}{\partial z'}z' - f = C$$

where C is a constant. Plugging in f, we get:

$$-\frac{z^2}{\sqrt{2(z')^2+z^2}} = C$$

Solving for z', we get:

$$z' = \sqrt{\frac{z^4}{2C} - \frac{z^2}{2}} = z\sqrt{\frac{z^2}{2C} - \frac{1}{2}} \quad \Rightarrow \quad \frac{z'}{z\sqrt{\frac{z^2}{2C} - \frac{1}{2}}} = 1$$

Solving the differential equation, we get that, when $z \geq 0$ (where k represents a constant), our optimal curve (given that it passes through our initial points):

$$\sqrt{2}(z\sqrt{C}) + k = \theta \quad \Rightarrow \quad z = \frac{1}{\sqrt{C}}\cos\sec\left(\frac{\theta - k}{\sqrt{2}}\right)$$

As seen with the previous two situations, the general strategy is to consider an alternative coordinate system (in both cases, polar coordinates) in order to reduce the number of variables in our integrand so that we can get the integrand into a form that allows for the Euler-Lagrange equation to be used. In both cases, converting to polar form allowed us to manipulate our integral into a form that allows us to use the Euler-Lagrange equation. However, polar form does not always yield a usable form, and we must consider other coordinate systems.

Geodesics on a Sphere

Now, consider a unit sphere, $x^2+y^2+z^2 = 1$, and two points on the sphere, (x_0, y_0, z_0) at t_0 and (x_1, y_1, z_1) at t_1. The arc-length between the two points is once again given by:

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

We then consider the substitutions from Cartesian coordinates to Spherical coordinates:

$$x = \cos(\theta)\cos(\phi) \quad \Rightarrow \quad \frac{dx}{dt} = -\sin(\theta)\cos(\phi)\frac{d\theta}{dt} - \cos(\theta)\sin(\phi)\frac{d\phi}{dt}$$

$$y = \sin(\theta)\cos(\phi) \quad \Rightarrow \quad \frac{dy}{dt} = \cos(\theta)\cos(\phi)\frac{d\theta}{dt} - \sin(\theta)\sin(\phi)\frac{d\phi}{dt}$$

$$z = \sin(\phi) \quad \Rightarrow \quad \frac{dz}{dt} = \cos(\phi)\frac{d\phi}{dt}$$

Our integral becomes:

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{d\theta}{dt}\cos(\phi)\right)^2 + \left(\frac{d\phi}{dt}\right)^2}\, dt = \int_{\theta_0}^{\theta_1} \sqrt{\phi + \left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

Now, let

$$f(\theta, \phi, \phi') = \sqrt{\phi + (\phi')^2}$$

We now apply the Euler-Lagrange Equation to f:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = \frac{\partial f}{\partial \phi}$$

Multiplying both sides by ϕ' gives:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\phi'\right) - \frac{\partial f}{\partial \phi}\phi' = 0$$

Since $\frac{\partial f}{\partial \theta} = 0$, our equation becomes:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\phi' - f\right) = 0$$

Integrating both sides (where C represents an arbitrary constant) gives:

$$\frac{\partial f}{\partial \phi'}\phi' - f = C$$

We plug f into our equation:

$$\left(\frac{\phi'}{\sqrt{(\phi)+(\phi')^2}}\phi'\right) - \sqrt{(\phi)+(\phi')^2} = -\frac{(\phi)}{\sqrt{(\phi)+(\phi')^2}} = C$$

Solving for ϕ', we get:

$$\phi' = \sqrt{\frac{(\phi)}{C} - (\phi)} \quad \Rightarrow \quad \frac{\phi'\sqrt{C}}{\cos(\phi)\sqrt{(\phi) - C}} = \frac{\phi'(\phi)\sqrt{C}}{\sqrt{1 - C(\phi)}} = 1$$

We assume that $0 < C < 1$. Integrating both sides (where k represents an arbitrary constant):

$$\arcsin\left(\tan\tan(\phi)\sqrt{C}/\sqrt{1-C}\right) = \theta + k \Rightarrow \tan\tan(\phi)\sqrt{C}$$

$$= \sqrt{1-C}\sin\sin(\theta + k)$$

Therefore, our optimal curve would be the one that passes through the initial points and satisfies the equation:

$$\tan\tan(\phi)\sqrt{C} = \sqrt{1-C}\sin\sin(\theta + k).$$

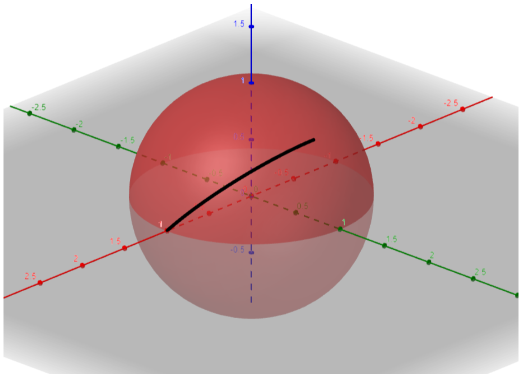

Fig. 3 Graph of the geodesic along the surface of the sphere

Applications

Previously, we have derived a way to find the optimal curve/function and applied that method to various aspects of pure mathematics. In this section, we will continue to use the previously derived equations and apply them to other topics. These topics include both finance and physics.

Finance

Depurkar[2] proposes a situation where a firm must produce n units of a good in T hours, and they would like to minimize the cost of production. The firm produces such that $p(t)$ represents the number of units produced at time $t \geq 0$, where t is represented in hours. This means that the change in the number of units produced at time t is equal to $p'(t)$. The firm has two sources of cost:

- One hour of storage costs a dollars per unit.
- Production at a rate of $p'(t)$ rises linearly with the production rate. That is, for every 1 unit increase in the production rate, the cost of production per unit increases by b dollars.

Both of these quantities can be expressed in terms of p: The cost from storage from time $t = 0$ to $t = T$ can be represented as:

$$\int_0^T ap(t)\,dt$$

The cost from production at a rate of $p'(t)$ from time $t = 0$ to $t = T$ can be represented as:

$$\int_0^T b(p'(t))^2\,dt$$

This means that the total cost can be represented as:

$$\int_0^T ap(t) + b(p'(t))^2\,dt$$

To minimize the cost, we find the function $p(t)$ that minimizes the value of the integral given the initial conditions that $p(0) = 0$ and $p(T) = n$. To do this, we apply the Euler-Lagrange Equation to the function $f(t, p, p') = ap(t) + b(p')^2$:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial p'}\right) = \frac{\partial f}{\partial p} \implies \frac{d}{dt}(2bp') = a \implies 2bp' = at + C_1$$

where C_1 represents an arbitrary constant. Solving this differential equation, we get that:

$$p'(t) = \frac{at + C_1}{2b} \implies p(t) = \frac{at^2}{4b} + \frac{C_1 t}{2b} + C_2$$

where C_2 represents an arbitrary constant. We plug in our initial conditions of $p(0) = 0$ and $p(T) = n$, and we get that $C_1 = \frac{2b}{T}\left(n - \frac{aT^2}{4b}\right)$ and $C_2 = 0$, getting us the equation:

$$p(t) = \frac{at^2}{4b} + \frac{1}{T}\left(n - \frac{aT^2}{4b}\right)t$$

Although not every production function will be as described, there are many production functions that will ultimately result in an integral with a first-order differential equation as the integrand. In these situations, the Euler-Lagrange equation can still be used, and the resulting differential equation that results can be solved to yield the optimal production function.

Physics

Consider two points, $P : (x_1, 0)$ and $Q : (x_2, y_2)$ with $y_2 < 0$. A bead with mass m starts at rest and travels along a frictionless curve that connects the two points. We would like to find the optimal curve, $y(x)$, such that the time the bead takes to travel from P to Q is minimized. Simmons[1] notes this as the well-known Brachistochrone Problem proposed by Johann Bernoulli in 1696. This situation is notable due to the common physical conditions that are of interest, such as gravity and conservation of mechanical energy. Additionally, time optimization is of great importance, and we show here how the Calculus of Variations can be used to better optimize time scenarios.

To solve this, we use the basic principle that time equals distance divided by speed. Then, by Conservation of Mechanical Energy, we have that $U_0 + K_0 = U_t + K_t$, where U_k represents the potential energy at time k and K_k represents the kinetic energy at time k. Since $K_0 = 0$ and $U_0 - U_t = mgy$ (where y represents the distance traveled by the bead), we have that:

$$K_t = U_0 - U_t = mgy$$

Since $K = \frac{1}{2}mv^2$, where v is the velocity of the bead, the velocity of the bead is equal to $\sqrt{2gy}$. The distance that must be traveled at that instant can be found using the formula for arc length. Therefore, the total time it takes for the bead to travel along the path is equal to:

$$\int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,dx$$

From there, let

$$f(x,y,y') = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}$$

and apply the Euler-Lagrange Equation:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

We multiply both sides by y', which gets us:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y' - \frac{\partial f}{\partial y}y' = 0$$

Since $\frac{\partial f}{\partial x} = 0$, we get that:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y' - \frac{\partial f}{\partial y}y' - \frac{\partial f}{\partial x} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}y' - f\right) = 0$$

We integrate both sides to get:

$$\frac{\partial f}{\partial y'}y' - f = c_1$$

where c_1 is a constant. Plugging in f, we have:

$$\frac{(y')^2}{\sqrt{2gy\sqrt{1+(y')^2}}} - \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} = \frac{1}{\sqrt{2gy}}\left(-\frac{1}{\sqrt{1+(y')^2}}\right) = c_1$$

Solving for y', we get that:

$$y' = \sqrt{\frac{1-2gc_1^2y}{2gc_1^2y}} = \frac{\sqrt{1-2gc_1^2y}}{c_1\sqrt{2gy}} \Rightarrow y'\frac{c_1\sqrt{2gy}}{\sqrt{1-2gc_1^2y}} = 1$$

$$\frac{1}{2gc_1^2}\left(\arcsin\left(c_1\sqrt{2gy}\right) - c_1\sqrt{2gy(1-2c_1^2gy)}\right) = x+k$$

where k represents an arbitrary constant.

Conclusion

The use of the Calculus of Variations to solve optimization problems, both within pure mathematics and otherwise, has become very prominent since the derivation of the Euler-Lagrange equation, as shown above. Our derivation of the equations for geodesics, in addition to the derivation of many optimal curves that are applicable to other fields of study, such as physics and finance, demonstrate the importance of the results obtained from the study of the Calculus of Variations, as well as further results that will come up in the future.

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References

1 G. Simmons, *Differential equations with applications and historical notes.CRC Press*.

2 A. Deopurkar, *Analysis and Optimization: Calculus of Variations*.