'The Pythagorean Theorem': A Comparison of Ancient Greek and Ancient Chinese Mathematics

Jun Yan Qiang

Received December 23rd, 2022
Accepted April 6, 2023
Electronic access June 15th, 2023

This paper aims to study the differences in the methodologies of ancient Greek and Chinese mathematics. It does so by examining Euclid's Elements and The Nine Chapters on the Mathematical Art, ancient mathematical texts that guided mathematical thinking in Greece and China, respectively, for centuries. Through comparing these treatises and studying the origin and nature of ancient Greek and Chinese mathematics, it is clear that the approach of ancient Greek mathematicians could not be any more distinct from that of their ancient Chinese counterparts. Such an occurrence is thoroughly demonstrated in this paper's discussion of the treatises' respective proofs of the Pythagorean theorem. Nonetheless, both civilizations' mathematical discoveries and contributions would play pivotal roles in shaping modern mathematics.

Introduction

It is widespread knowledge that the ancient Greek civilization contributed significantly to the development of modern mathematics. As the homeland to the likes of Euclid and Archimedes, the civilization has long been acknowledged for its invaluable involvement in the idea of formal proof in geometry, as well as number theory and combinatorics. However, another society's significant mathematical contributions that remain in contention with those of Greece have historically been disregarded and underappreciated. This civilization is ancient China, the source of a larger breadth of mathematical ideas and techniques than any other contemporaneous society[1]. The Chinese allowed algebra and geometry to develop substantially using their real number system with multiple numeral systems.

Interestingly, the mathematics of both civilizations first arose to fulfill practical purposes in order to maintain the survival of their respective societies. Nevertheless, it is compelling that while ancient Greek mathematics would advance to be known for its deductive foundations, ancient China became, on the contrary, noteworthy as a result of its "inductively-conceived mathematics"[2].

To study the fascinating differences between ancient Greek and Chinese mathematics, this paper will dive into the content of Euclid's Elements and The Nine Chapters on the Mathematical Art, two indispensable mathematical treatises for each civilization. As one of, if not the most important and successful mathematical textbooks in history, Euclid's teachings from the Elements still form the foundation of geometry taught in high school curricula today[3]. On the other hand, the Nine Chapters is one of the oldest and most impactful extant Chinese mathematical treatises, used until circa 1600 "as a basic textbook for mathematics"[4]. In particular, this paper will analyze both treatises' derivation of the Pythagorean theorem in hopes of uncovering the nature and basis that caused mathematics to develop the way it did in both ancient Greece and China.

Literature Review

The comparative study of ancient Greek and Chinese science and mathematics is a field that has been examined and discussed by several scholars in the past decades. To establish a basis to understand the civilizations' fundamental treatises, it is necessary first to examine the existing literature in this realm. Geoffrey Ernest Richard (G. E. R.) Lloyd is a leading historian in this field, having published and contributed to many books, edited volumes, and articles comparing ancient Greek and Chinese science, mathematics, and culture.

First and foremost, Lloyd authored The Ambitions of Curiosity, a book focused on confronting stereotypes about the culture and philosophy of ancient Greece and China. Lloyd emphasized the influence of sociopolitical circumstances on the development of the civilizations' scientific inquiry and method. In the chapter "The Number of Things," Lloyd inquires into why mathematics evolved divergently in these ancient civilizations. Lloyd furthers the argument regarding the pre-modern belief, especially upheld by the Greeks, of the presence of mathematical numbers in nature, such as in the movement of planets[5]. He concluded that the Greeks used this belief alongside axiomatic methods to ensure certainty in their mathematical arguments. He also elaborates that the Chinese mainly sought cohesion within the various strands of mathematics and focused less on mathematical deduction based on axiomatic bases and achieving “incontrovertibility”[6].

Moreover, the book Ancient Greece and China Compared, co-edited by Lloyd, is a literary piece that serves as a groundbreaking interdisciplinary study of both civilizations. Notably, a former student of Lloyd and now a prominent historian of ancient mathematics, Reviel Netz, investigates ancient Greek and Chinese mathematics in his article “Divisions, Big and Small: Comparing Archimedes and Liu Hui.” Netz explores the differences between ancient Chinese and Greek mathematics by studying how these civilizations approached mathematical proofs and contrasts regarding the types of mathematical objects they examined. His investigation is conducted by collating how influential mathematicians of both civilizations studied the Box-Lid, the resulting intersection when two cylinders are circumscribed within a single cube. Utilizing observations and analysis from the Nine Chapters and Euclid’s Elements, Netz concludes that the differences in the methods of approach are noticeable and can be cited to influence and have been influenced by the respective politics, society, and governance of both civilizations[7].

Historians have also noted the influence of the social-political aspect of civilizations on their mathematics. This is done so, for instance, in Alexei Volkov’s article “Argumentation for state examinations: demonstration in traditional Chinese and Vietnamese mathematics.” In this article, he investigates commentaries on ancient Chinese mathematical texts to examine the impact of the circumstances behind the commentaries’ production and use on how they were interpreted. While historians have considered these commentaries as professionally written “purely mathematical texts,” Volkov shows that the case is not so simple[8]. In particular, he argues that these mathematical commentaries shaped the standards for the “style and structure” of traditional mathematics within other institutes during the first millennium since they set the guidelines of the curricula of state-sponsored educational institutions such as the Mathematical College in Chang’an[9]. Furthermore, in their literary piece “Sing, Muse, of the Hypotenuse: Influences of Poetry and Rhetoric on the Formation of Greek Mathematics,” Apostolos Doxiadis and Michalis Sialaros examine the means in which political rhetoric influenced ancient Greek mathematics. They note that social life and the political conditions of Athens and other states greatly contributed to the distinctions between ancient Greek and non-Greek mathematics. Particularly, the significant logical element of Greek-style mathematics would be deeply rooted in the Greek’s usage of rhetoric for debates and judicial procedures[10].

Historical Context

Before analyzing Euclid’s Elements and the Nine Chapters, it is necessary to investigate how traditional mathematics developed in ancient Greece and China and the roles they played in their respective societies.

While the oldest extant complete Greek mathematical text is accredited to Autolycus of Pitane circa 330 BCE, the ancient Greek mathematics known today was created around 600 - 330 BCE, a time from which no surviving texts exist[11]. Many Greek scholars that arose from that point onwards are recognized for their discovery of and work in numerous aspects of arithmetic and geometry, including conic sections and solid geometry. Nevertheless, despite their essential contributions to mathematics, they have been portrayed as “philosophers,” “sophists,” or “astronomers.” Such an occurrence can be attributed to the following phenomena.

Firstly, the Ancient Greek word máthēma, which is derived from the word mathēmatikós, meaning mathematical, initially referred precisely to “that which is learned”[12]. Hence, a mathematician was merely a “person who is fond of learning,” which justified ancient Greek scholars’ equal if not greater renown for their role in other realms[13]. Additionally, according to Asper, the profession of mathematics in ancient Greece was very distinct from modern mathematics. Mathematics today has become an occupation, whereas ancient Greek-style mathematics was seen as a “form of game”[14].

It should also be noted that ancient Greek texts from the period of Hellenistic Egypt contain practical mathematics of the same type as demotic Egyptian texts written approximately in the same period[15]. This resulted from the fact that the ancient Greeks looked highly upon Egyptian mathematics and believed that their ancestors learned the art of mathematics from the Egyptians. Traditional ancient Greek mathematics would nonetheless be reasonably distinct from traditional Egyptian mathematics. While the ancient Egyptians utilized solution-centric problem-solving to perform practical calculations in areas such as accounting and architecture, the ancient Greeks focused heavily on deductive methodology written in general abstract terms. Political and legal rhetoric also greatly influenced Greek-style mathematics. Lloyd argues that the development of rigorous arguments in Greek philosophy and mathematics must be seen against the backdrop of rhetoric[16]. Additionally, Doxiadis and Sialaros affirm that Greek rhetors used techniques widely present in archaic poetry[17]. Lloyd also shows that there were many intellectual pursuits where presentation was heavily influenced by a public, oral presentation akin to a political speech[18].

Chinese traditional mathematics had a very different role in society. First and foremost, Chinese mathematics is deeply rooted in mysticism. It is said that the legendary founder of the Xia Dynasty, Emperor Yu, once received a diagram known as Lo Shu from a Lo River tortoise. This diagram supposedly embodied the principles of Chinese mathematics, and depictions of Yu's interaction with the tortoise have ornamented Chinese mathematics texts for centuries[19]. Due to Yu's mythical monumental role in developing flood control systems in his lifetime, Chinese science and mathematics were used for hydraulic engineering in their early stages[20].

Later on, Chinese mathematics would develop to be significantly more extensive. Mathematics would be established in practical aspects of society, such as land measurement in agriculture[21]. Though certain matters, such as Liu Hui's meticulous estimation of the value of pi, suggest otherwise, it is generally argued that the ancient Chinese halted their exploration of mathematics beyond practical interest[22]. Chinese geometry was based on handling tangible and visualizable elements and generalizations. It also favored demonstrating theorems and techniques using examples, which led mathematical texts to hold many engaging problems regarding real-life scenarios. Chinese mathematics was also bureaucratic in the sense that it was very connected to the government. Mathematical texts by ancient Chinese mathematicians and scholars were used in the curricula of government-sponsored education institutions[23].

Synopses of the Treatises

Euclid's Elements

Euclid's Elements is a mathematical treatise written circa 300 BCE, during the Hellenistic period of ancient Greece. The treatise is comprised of 13 books, each centered around distinct subtopics within mathematics. Historically, these 13 books, written in ancient Greek, have been roughly categorized into three sections: plane geometry (Books I-VI), arithmetic (VII-X), and solid geometry (XI-XIII)[24].

The author of this treatise, Euclid, was a Greek mathematician alive during the third and fourth centuries BCE. Beyond his residence in Alexandria around 300 BCE, modern historians know little about the life of Euclid[25]. He is often known as the father of geometry for his contributions to the branch of mathematics, primarily through this treatise. Nonetheless, the books of the Elements primarily consist of theorems and knowledge discovered by Euclid's Greek mathematical predecessors rather than himself, such as Pythagoras, Theaetetus of Athens, and Eudoxus of Cnidos[26]. Moreover, modern historians do not have sizable knowledge of the lives of these ancient Greek mathematicians, many of whom they are principally familiar with through myths and legends.

Euclid's significant contribution was creating a systematic and exhaustive collection of existing knowledge in the field, organized into a logical sequence that is the Elements. Hence, it represented Greek mathematical mastery and understanding up until the writing of the treatise. Beyond its original purpose, Euclid's Elements became the comprehensive heart of mathematical teaching for thousands of years. Its method of approach is known as the axiomatic system. This technique involved establishing several definitions, postulates, and common notions at the beginning of Book I. The 23 definitions introduce the reader to terms such as points, lines, surfaces, and figures. On the other hand, the five postulates were axioms – self-evident geometrical truths recognized without proof. While the five common notions were also axioms, they were general assumptions that concerned mathematics beyond geometry. All subsequent theorems and outcomes described in the Elements are achieved through coherent deduction based on such postulates. Most succeeding books would also open with sets of definitions, summing up to 131 definitions across the treatise. The Elements is the first prominent example of the use of this method, which would be known as Euclidean geometry. The treatise also emphasized the necessity of formal proofs or derivation – convincing arguments to validate theorems and ideas – through its propositions, demonstrated with singular logical progressions. The notion of formal proofs originated from the concept of deductive reasoning, the process of logically drawing conclusions from several premises.

Additionally, each book covered numerous propositions written based on the works of previous mathematicians. It must also be noted that the 465 propositions throughout the 13 books are often built off one another. For instance, Euclid first had to construct a square in proposition I.46. Only then could he construct a figure which inscribes squares on the sides of a right triangle to prove the Pythagorean theorem one proposition later. Furthermore, the geometric construction featured were components of the Elements' constructive approach to mathematics, which involved proving the existence of geometric shapes used in propositions with a straightedge and a compass[27]. Such geometric figures were presented with diagrams, and subsequent deductions and statements were made about such diagrams.

The Nine Chapters on the Mathematical Art

Jiu Zhang Suan Shu, known as The Nine Chapters on the Mathematical Art or the Nine Chapters in English, is an ancient mathematical book from China. Written in Classical Chinese, the Nine Chapters contains writings dating back to at least 1100 BCE, centuries before the rise of ancient Greek mathematics[28] It is generally recognized and accepted to have been assembled into a book during the Eastern Han dynasty, which took place between 25 and 220 CE[29].

Various ancient Chinese scholars compiled this treatise over several generations, including, most notably, Han dynasty imperial ministers Zhang Cang during the second century BCE and Geng Shou Cang a century later[30]. These scholars' contributions were efforts to reinstate lost Chinese classics. More-

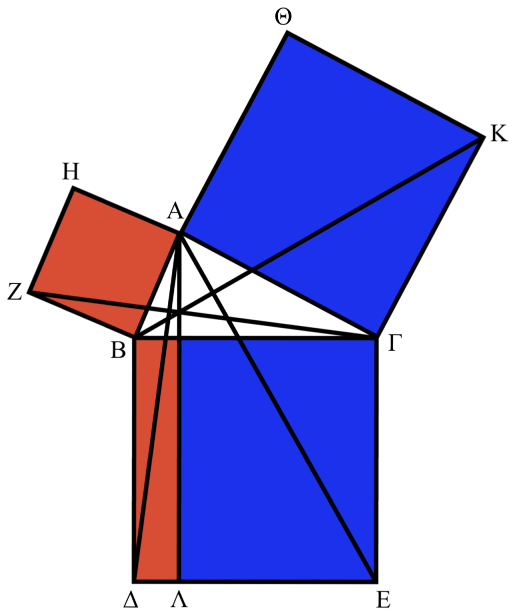

Fig. 1

over, the book is also famously affiliated with Liu Hui – a Han dynasty mathematician – and his commentary on the Nine Chapters in 263 CE for two primary reasons. Firstly, in his commentary, Liu makes significant contributions through his arguments regarding the book's content, as the Nine Chapters' rules and algorithms initially lacked formal proofs[31]. Most notable is Liu Hui's extraordinary approximation of the value of pi in his commentary. Secondly, Liu began a tradition of commenting on the book that was repeated by later scholars such as Li Chung Feng in the Tang Dynasty[25].

The Nine Chapters was written in a question-answer format in which it introduces 246 math problems covering arithmetic, algebra, and geometry, divided into nine chapters. Each chapter was loosely named, usually in accordance with the first problem of the chapter. For instance, Chapter One is named fang tian (rectangular field), and Chapter Two is named su mi (millet and rice), both of which are subjects of each chapter's first problems. These problems were believed to be passed down by generations of Chinese people[26].

The ancient Chinese numeral system, known as the rod numeral system, consisted entirely of the manipulation sticks on a counting board. This system was the foundation for the basics of arithmetic and the development of mathematics in the country[24]. It allowed the Chinese to invent methods for addition, subtraction, multiplication, and division. As a result, it accelerated the development of the ancient Chinese civilization through a plethora of social and economic aspects of the nation. The rod numeral system was essential and heavily present in this treatise. The book mainly utilizes general methods in which it states a technique of mathematical approach in straightforward language, accompanied by several problems illustrating the method. Problems were followed by their answer and solution method, generally without a justification or explanation for such a method[3]. The Nine Chapters' written methods could only describe the placement of specific rod numerals to perform operations or known mathematical techniques[27]. Consequently, such descriptions were frequently condensed or excluded.

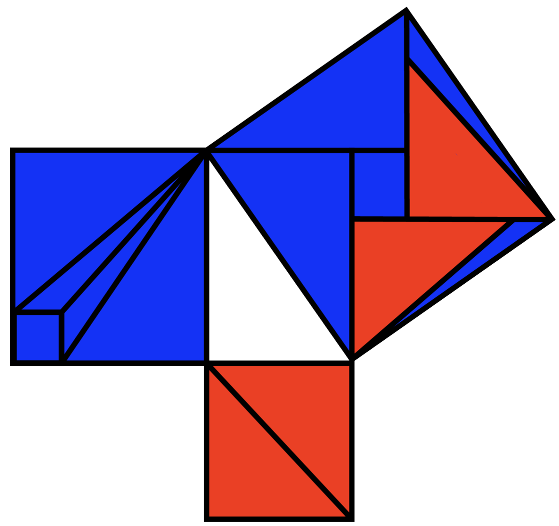

Fig. 2

The Pythagorean Theorem

The Pythagorean theorem, also known as Pythagoras' theorem, is a geometric theorem attributed to and named after the ancient Greek philosopher Pythagoras. In modern terms, the theorem states that the sum of the squares of the legs of a right triangle is equivalent to the square of the triangle's hypotenuse. This mathematical relationship appears in Euclid's Elements and the Nine Chapters, which provide extensive proof of the theorem. Before conducting a comparative study of the proofs, it should be noted that plenty of evidence suggests that the Pythagorean theorem was discovered long before Pythagoras' lifetime. Modern scholars realize that Babylonian tablets, dating back to the Old Babylonian Empire (between 1900 BCE and 1600 BCE), show that the Babylonians were familiar with the mathematical relationship through Pythagorean triples[28].

In Euclid's Elements, Euclid uses a straightedge and a compass to prove the theorem in Proposition 47 of Book I as he does in his other proofs. He states, "In right-angled triangles, (the area of) the square on the side subtending the right angle is equal to the (sum of the areas of the) squares on the sides containing the right angle"[29]. The proposition commences by constructing a square on each side of the right triangle, which has been proved possible in Proposition 46. After constructing multiple auxiliary lines parallel to certain segments and from point to point, Euclid splits the square on the hypotenuse into two different parallelograms with auxiliary line $A\Lambda$. Euclid then attempts to prove that the area of the square on the shorter leg is equivalent to the area of the smaller parallelogram and that the area of the square on the longer leg is equivalent to the area of the larger parallelogram. He begins by proving the equivalence in area between multiple triangles created by the auxiliary lines using previously established properties from the Elements. In the end, he proves the equivalence in area of both squares ZHAB and $\Theta K\Gamma A$ with their corresponding parallelograms of the same color within square $B\Gamma E\Delta$, as visualized in Figure 1, deriving the Pythagorean theorem.

In China, the Pythagorean theorem is known as the Gou-Gu theorem, with the shorter leg of the right triangle called the gou, the longer leg called the gu, and the hypotenuse called the xian. Chapter nine of the Nine Chapters discusses the Gou-Gu theorem and begins with three problems asking for the third side of a right triangle using the theorem. According to the treatise, the theorem method is to "Square each of gou and gu, add and extract the square root to obtain xian"[30]. Though original descriptions of the proof of the theorem are not extant, we can still obtain the treatise's general deduction through Liu Hui's commentary. Liu Hui's explanation of the Gou-Gu is translated as follows:

The shorter leg multiplied by itself is the red square, and the longer leg multiplied by itself the blue square. Let them be moved about so as to patch each other, each according to its type. Because the differences are completed, there is no instability. They form together the area of the square on the hypotenuse; extracting the square root gives the hypotenuse[31].

While the diagram that Liu Hui describes is lost, it is evident that Figure 2 is a close interpretation of the cutting up and shuffling of the squares on gou and gu. This method is standard in Liu Hui's commentary and, at first glance, does not prove the theorem to be valid due to its lack of substantial statements and reasons that would be required in the more traditional geometric proofs, such as Euclid's derivation of the theorem. However, upon deeper inspection, it is clear that Liu Hui's abstract approach utilizing correct mathematical concepts to demonstrate the Gou-Gu theorem.

Conclusion

When considering both proofs of the Pythagorean theorem together, it is apparent that Euclid's abstract axiomatic framework in his method heavily contrasts the Chinese's concrete approach by rearranging the areas. Euclid's argument is based on definitions, axioms, and theorems about similar triangles and areas that he progressively introduces throughout Book I. He then proves the theorem with a long paragraph recording the process. On the contrary, the Nine Chapters immediately introduces its straightforward derivation of the Gou-Gu theorem after providing practical problems.

Both lines of reasoning directly reinforce previously established perceptions of both civilizations' styles of mathematics. While the Greeks emphasized the use of deductive reasoning and conclusive proofs that validated propositions and premises in their mathematical arguments, the Chinese focused on general mathematical methods that largely fulfilled practical interests. In the end, the reflection of these mathematical styles in both mathematical treatises would seem almost expected, given the nature and origin of both ancient Greek and Chinese mathematics. Nonetheless, it is worth noting the considerable differences between the mathematical methodologies of both civilizations – which are rooted in the same practical origins – that highlight the absolute beauty of mathematics in the ancient and modern world.

References

1 F. Swetz, *Mathematics Magazine*, 1979, **52**, 10–19.
2 R. Hartshorne, *Geometry: Euclid and beyond*, Springer Science & Business Media, 2013.
3 R. Schwartz, *The Right Angle*, 2008, **16**, 8–12.
4 G. E. R. Lloyd, *The ambitions of curiosity: Understanding the world in ancient Greece and China*, Cambridge University Press, 2002, vol. 64, p. 49.
5 G. E. R. Lloyd, *The ambitions of curiosity: Understanding the world in ancient Greece and China*, Cambridge University Press, 2002, vol. 64, p. 67.
6 R. Netz, *Divisions, big and small: Comparing Archimedes and Liu Hui. Ancient Greece and China Compared*.
7 A. Volkov, *The History of Mathematical Proof in Ancient Traditions*, vol. 509.
8 A. Volkov, *The History of Mathematical Proof in Ancient Traditions*, vol. 536.
9 A. Doxiadis and M. Sialaros, *Markus Asper (ed.)*, 2013, 375.
10 A. Doxiadis and M. Sialaros, *Markus Asper (ed.)*, 2013, 372.
11 A. Doxiadis and M. Sialaros, *Markus Asper (ed.)*, 2013, 374.
12 M. Asper, *The Oxford Handbook of the History of Mathematics*, vol. 123.
13 A. Doxiadis and M. Sialaros, *Markus Asper (ed.)*, 2013, 368.
14 G.E.R.Lloyd, *Magic, Reason and Experience*, 1979, **246**,.
15 A. Doxiadis and M. Sialaros, *Markus Asper (ed.)*, 2013, 390.
16 G.E.R.Lloyd, *Magic, Reason and Experience*, 1979, **86**,.
17 F. Swetz, *Mathematics Magazine*, 1979, **52**, 10–19.
18 J. W. Dauben, *Amphora: Festschrift für Hans Wussing zu seinem 65. Geburtstag Festschrift for Hans Wussing on the Occasion of his 65th Birthday*, 1992, 133–155.
19 L. Yong, *Archive for History of Exact Sciences*, **47**, 146.
20 A. Volkov, *The History of Mathematical Proof in Ancient Traditions*, vol. 510.
21 B. Artmann, *Apeiron: A Journal for Ancient Philosophy and Science*, **24**, 2–3.
22 R. Fitzpatrick, *Euclid's Elements of Geometry*.

23 R. Hartshorne, *Geometry: Euclid and Beyond*.
24 L. L. Yong, *Archive for history of exact sciences*, 1994, 47.
25 J. Dauben, *Amphora*, 1992, **135**,.
26 M.-K. Siu, *Educational Studies in Mathematics*, **24**, 346.
27 L. L. Yong, *Archive for history of exact sciences*, 1994, 47.
28 D. Mansfield, *Foundations of Science*, **26**, 977.
29 R. Fitzpatrick, *Euclid's Elements of Geometry*.
30 L. Yong, *Archive for History of Exact Sciences*, **47**, 39.
31 Q. Baocong, *Suan jing shi shu (Ten mathematical treatises*.

